A “Periodic Table” of Topological Orders

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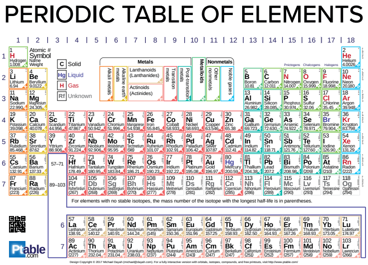
Periodic Table?



- Understanding phases of matter and phase transitions is a key question in statistical physics and condensed matter physics.
- Is it possible to have a classification of phases of matter as beautiful as the Periodic Table of chemical elements?
- STEP ONE: NAME THEM!

Warm Up! Topology

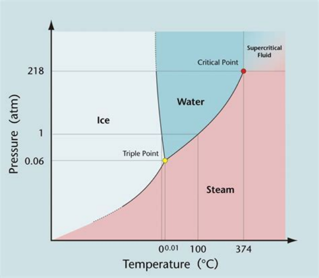
Topology tells some difference between a ball and a donut.
Intuitively, a donut has a hole while a ball doesn't.

Topology

- Topology studies the properties under continuous deformations.
- Discontinuous deformation: cut, tear up, glue,...
- Topological invariant: quantities that is invariant as long as there is no discontinuous deformation.
- The number of holes is a topological invariant, genus:
    - A sphere (surface of a ball) has genus 0;
    - A torus (surface of a donut) has genus 1.

How is the idea of topology applied to physics?

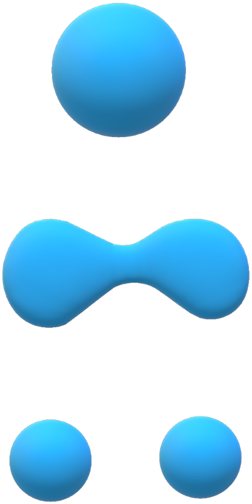
Phase and Phase Transition



A most traditional way to represent phases and phase transitions is the phase diagram. But it has limitations:

- Difficult to show more than two parameters.
- No direct description of phases. Difficult to have a simple "table" or "list" of phases.

“Topology” of Phases of Matter



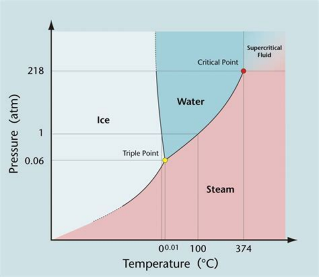
Treat a physical system like a “topological space” (hyper-surface, manifold,. . .)

- Continuous deformation: small local perturbations
- Discontinuity: phase transition
  gap closing for gapped phases
- “Topological invariants” of physical systems: quantities that stay invariant as long as there is no phase transition
- Topological invariants are direct description of phases. (THE NAME)

More is Different

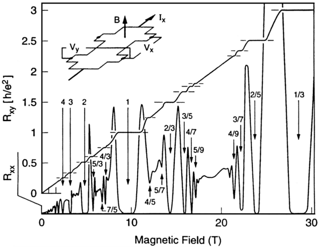
- Topological invariants (universal properties) are the observables of many-body system at the largest length (lowest energy) scale.
- Topological invariants are not a bunch of isolated quantities. They are correlated and structured.
- Enormous rich structures emerge at this scale.
- New paradigm may be needed.
- New mathematical languages have been introduced.

Symmetry



- The first topological invariant is the symmetries of physical systems.
- Ice has smaller symmetry comparing to water/steam. The phase transition is accompanied by symmetry breaking.
- Landau's symmetry breaking theory was once thought to describe all phase transitions. This would be true if symmetry was the *only* topological invariant.

Fractional Quantum Hall Effect



Horst Störmer and Daniel Tsui, 1982

- Phase transitions happen while symmetry remains the same.
- Imply topological invariants beyond symmetry.
- A most important feature in FQHE is the fractionalized quasiparticle excitations, or anyons.
- Have topological order

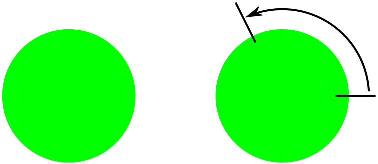
X.-G. Wen, Int. J. Mod. Phys. B 4, 239 (1990)
X.-G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990)

Symmetry as the *Trivial* Topological Invariant

- It is not conventional to call symmetry topological invariants.
- Phases whose topological invariants are only symmetries are *not* called topological phases. They are symmetry breaking phases,
- However, we can view symmetry as the *trivial* topological invariant, such that the extra topological invariants, e.g. in FQHE, are natural generalization of symmetry.

Symmetry as the *Trivial* Topological Invariant

- Group theory is the traditional language for symmetry.
- However, group itself describes “change”, not “invariants”.
- Symmetry transformations (elements in the group) are not observables, unless we break the symmetry.



- If a system is perfectly symmetric, the group is not physical (not measurable). Such paradox implies that a group is not the most natural language to describe a symmetry.

Symmetry as the *Trivial* Topological Invariant

- Noether's theorem: continuous symmetry ⇔ conserved quantities
- Tanaka duality: group ⇔ group representations
- The conserved quantities or group representations are the observables or invariants.

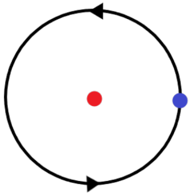
Symmetry transformation	Invariants
Translation	Energy or momentum
Rotation	Angular momentum
$U(1)$	Particle number
Z_2^f	Fermion number parity
G	$\mathrm{Rep}(G)$, symmetry charges
$Z_2^f \subset G^f$	$\mathrm{sRep}(G^f)$

So we should take $\mathrm{Rep}(G)$ or $\mathrm{sRep}(G^f)$ as the topological invariant. But in what sense is it *trivial*?

Topological Invariants beyond Symmetry

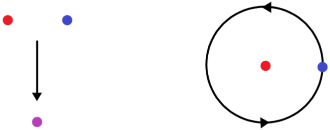
The most important topological invariants beyond symmetry is the statistics of emergent excitations.

Statistics refers to fusion generalized addition describing how the quantities are "conserved" and braiding:



Symmetry is the *trivial* invariant in the sense $\mathrm{Rep}(G)$ or $\mathrm{sRep}(G^f)$ has the trivial braiding statistics, namely Bose or Fermi statistics.

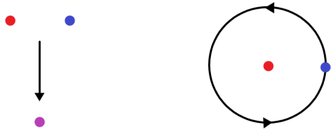
Topological Invariants beyond Symmetry



Local excitations (non-topological)

- The double braiding gives phase factor 1 (namely they are bosons or fermions).
- But they may have different symmetry charges, thus have non-trivial fusion properties (different conserved quantities).
- These are described by $\mathrm{Rep}(G)$ or $\mathrm{sRep}(G^f)$. Fusion is given by the tensor product of group representations.

Topological Invariants beyond Symmetry



Fractionalized excitations/anyons (topological)

- The double braiding gives a phase factor other than 1, or even not a phase factor.
- Also non-trivial fusion properties.
- Details are different in different dimensions.
- Tensor category and higher category theory kicks in.

Topological Invariants beyond Symmetry

There are still other topological invariants. For example, in the topological insulator, there is no fractionalized excitations like in the FQHE (no topological order). But it has symmetry protected gapless (conducting) edge states, distinguished from the trivial insulator state with the same global symmetry.

$\rightarrow$ Symmetry protected topological (SPT) phases

X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G Wen, Phys. Rev. B 87, 155114 (2013), Science 338, 1604 (2012)

Moreover, symmetry can have non-trivial interplay with topological order, known as symmetry enriched topological (SET) phases.

Fortunately, SPT/SET invariants can still be encoded into statistics. This is achieved by gauging the onsite symmetry, and we have the gauged symmetry fluxes as extra excitations, whose fusion and braiding with the original excitations encode SPT/SET invariants.

Topological Phases in Different Dimensions

	1+1D	2+1D	3+1D
Symmetry breaking	✓	✓	✓
SPT	✓	✓	✓
Topological order	×	✓	✓
???	×	×	✓

Complete classification in 1+1D

- Symmetry breaking phases $G \subset G_H$
  G_H is the symmetry group of the Hamiltonian
  G is the symmetry group of the ground states
- 1+1D Topological phases
  $G \subset G_H, \mathrm{pRep}(G)$ (or $H^2(G, U(1))$)
  Symmetry breaking, SPT

X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 83, 035107 (2011)
N. Schuch, D. Perez-Garcia, and I. Cirac, Phys. Rev. B 84, 165139 (2011)

2+1D Topological Phases

2+1D Topological Phases (in colloboration with Liang Kong and Xiao-Gang Wen)

PRB 94, 155113 (2015),1507.04673; PRB 95, 235140 (2017), 1602.05946; CMP 351, 709–739 (2017), 1602.05936

$$G \subset G_H, \quad \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, \quad c$$

$G \subset G_H$ – Symmetry breaking

$\mathcal{E}, \mathcal{C}, \mathcal{M}$ – unitary braided fusion categories (UBFC)
fusion and braiding (statistics) of quasiparticles (anyon model)

- $\mathcal{E}$ local excitations carrying group representations
  symmetric fusion category, $\mathrm{Rep}(G)$ or $\mathrm{sRep}(G^f)$
- $\mathcal{C}$ $\mathcal{E}$ plus "anyons", all bulk excitations
  UBFC with Müger center $\mathcal{E}$, $\mathrm{UMTC}_{/\mathcal{E}}$
- $\mathcal{M}$ $\mathcal{C}$ plus "gauged symmetry fluxes", excitations in the "gauged" phase
  minimal modular extension of $\mathcal{C}$

c – chiral central charge, to address "invertible" stack to trivial states

Table of 2+1D Topological Phases

List $\mathcal{C}$ in terms of anyon spectrum:

$\mathbb{Z}_2$ symmetry

$N_c^{\lvert\Theta\rvert}$	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$
$2_0^{\zeta_2^1}$	$1, 1$	$0, 0$
$3_2^{\zeta_2^1}$	$1, 1, 2$	$0, 0, \frac{1}{3}$
$3_{-2}^{\zeta_2^1}$	$1, 1, 2$	$0, 0, \frac{2}{3}$
$4_1^{\zeta_2^1}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, \frac{1}{4}$
$4_1^{\zeta_2^1}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, \frac{1}{4}$
$4_{-1}^{\zeta_2^1}$	$1, 1, 1, 1$	$0, 0, \frac{3}{4}, \frac{3}{4}$
$4_{-1}^{\zeta_2^1}$	$1, 1, 1, 1$	$0, 0, \frac{3}{4}, \frac{3}{4}$
$4_{14/5}^{\zeta_2^1}$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{2}{5}, \frac{2}{5}$
$4_{-14/5}^{\zeta_2^1}$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, 0, \frac{3}{5}, \frac{3}{5}$
$4_0^{\zeta_2^1}$	$1, 1, 2, 2$	$0, 0, \frac{1}{5}, \frac{4}{5}$
$4_4^{\zeta_2^1}$	$1, 1, 2, 2$	$0, 0, \frac{2}{5}, \frac{3}{5}$

$\mathbb{Z}_2^f$ symmetry "fermion phases with no symmetry"

N_c^F	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$
2_0^F	$1, 1$	$0, \frac{1}{2}$
4_0^F	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$
$4_{1/5}^F$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$
$4_{-1/5}^F$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$
$4_{1/4}^F$	$1, 1, \zeta_6^2, \zeta_6^2$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$
6_0^F	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$
6_0^F	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$
$6_{1/7}^F$	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{5}{14}, -\frac{1}{7}, -\frac{3}{14}, \frac{2}{7}$
$6_{-1/7}^F$	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{5}{14}, \frac{1}{7}, \frac{3}{14}, -\frac{2}{7}$
6_0^F	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{1}{2}$
6_0^F	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2}$

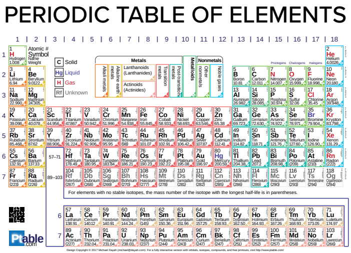
$$\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$$

N – number of anyon types; "rank"

d_i – quantum dimension "internal degrees of freedom"

s_i – topological spin "internal angular momentum mod 1"

Relations Between Topological Phases



- STEP TWO: ORGANIZE THEM!
- There are several ways to relate topological phase, such as layer-construction (stacking), condensation, hierachy construction, . . .

Make the table periodic!

Boson Condensation

Condense bosons (self-bosonic anyons) into the trivial state:

- The condensed anyon must satisfy centain constraints.
  Condensable algebra and its representation theory in UBFC

  S. Eliëns, Anyon condensation, University of Amsterdam MSc. Thesis
  Review: L. Kong, Anyon condensation and tensor categories, Nuclear Physics B 886 (2014)
- Symmetry breaking (condensing a usual, not anyonic, boson, namely boson in $\mathcal{E}$) is a special case.
- Group topological phases into Witt classes:
  If Phase A becomes Phase B after a boson condensation, A and B can be connected by gapped domain wall, and they belong to the same Witt class.
- A simple theory of the gapped domain wall based on the tunnelling matrix with integer entries.

  T. Lan, J. C. Wang, and X.-G. Wen, PRL 114, 076402 (2015), arXiv:1408.6514.

Generalized Hierachy Construction

Generalize the hierachy construction of quantum hall states.

F. D. M. Haldane, PRL 51, 605 (1983); B. I. Halperin, PRL 52, 1583 (1984).

In any topological order, let Abelian anyons form Laughlin-like states:

- Preserves symmetry and non-Abelian properties the potential of braiding based topological quantum computation.
- Reversible.
- Group topological phases into Non-Abelian Families.
- All Abelain topological orders form the trivial non-Abelian family.
- Each family has root phases with the smallest rank. Abelian anyons in the roots have trivial self and mutual statistics among them (form a symmetric fusion category).

T. Lan and X.-G. Wen, PRL 119, 040403 (2017), arXiv: 1701.07820
T. Lan, PRB 100, 241102(R) (2019), arXiv:1908.02599

List phases in terms of topological invariants in a Periodic Table.

“Periodic Table”

Abelian Anyons in Roots

$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2^f$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2^f$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\cdots$
Abelian	$SU(2)_{4k}$	$SU(3)_3$	$SU(2)_2$ Ising	$SU(2)_{4k} \boxtimes SU(2)_{4k'}$	$\frac{U(1)_2}{\mathbb{Z}_2}$ Ising⊠Ising	$D(D_4) \simeq D^{\alpha_1}(\mathbb{Z}_2^3)$	$\vdots$
$SO(3)_3$ Fib	$SO(5)_2$	$SU(3)_6$	$SU(2)_{4k-2}$	$\frac{U(1)_{4k}}{\mathbb{Z}_2}$	$\frac{U(1)_{4k+2}}{\mathbb{Z}_2}$	$D(Q_8) \simeq D^{\alpha_2}(\mathbb{Z}_2^3)$	$\vdots$
$SO(3)_{2k-1}$	$D(S_3)$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$D^{\alpha_3}(\mathbb{Z}_2^3)$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Roots of Non-Abelian Families

Summary and Outlook

	1+1D	2+1D	3+1D
Symmetry breaking	✓	✓[1]	✓
SPT			✓
TO with symmetry (SET)	$\times$		✓
Topological order	$\times$		✓[2]
Fracton phase	$\times$	$\times$	✓
???	$\times$	$\times$	???

2+1D Topological Phases with Symmetry[1]

$$G \subset G_H, \quad \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, \quad c$$

3+1D Topological Order[2]

PRX 8, 021074 (2018), arXiv:1704.04221; PRX 9, 021005 (2019), arXiv:1801.08530.

Gauging: 3+1D SPT $\twoheadrightarrow$ 3+1D topological order

3+1D topological orders are all gauged SPTs.