Structure coefficients of condensable algebra

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Shenzhen Strongly Correlated Forum, Jan 16, 2020

Outline

- Condensable algebra is the key mathematical notion for anyon condensation.
- I will describe the standard prescription to translate notions in fusion category into concrete numbers/matrices/tensors and equations between them. Generalized tensor network
- Following such prescription, one can extract the structure coefficients of condensable algebra.
- Physically, structure coefficients can be observed/measured by the overlap between groundstates of condensed and uncondensed phases, on punctured sphere or higher genus surfaces. TL, Xueda Wen, Liang Kong, and Xiao-Gang Wen, arXiv:1911.08470.

Rewrite set-theoretical notions in terms of maps

- Set-theoretical notions are usually written in terms of elements and equations.
- In category theory, there are no elements; objects are thought as black boxes and no internal structures are assumed.
- The only things one can manipulate are the maps or morphisms between objects.
- So the first step is to rewrite set-theoretical notions in terms of maps. Get rid of elements

Rewrite set-theoretical notions in terms of maps

For example, a semi-group G is a set G with a binary operation

$$\begin{aligned} G \times G &\to G, \\ (a, b) &\mapsto ab, \end{aligned}$$

which is associative. In terms of elements:

$$(ab)c = a(bc).$$

Rewrite set-theoretical notions in terms of maps

For example, a semi-group G is a set G with a binary operation

$$\begin{aligned} m : G \times G &\to G, \\ (a,b) &\mapsto ab \equiv m(a,b), \end{aligned}$$

which is associative. In terms of elements:

$$(ab)c = a(bc), \quad m(m(a,b),c) = m(a,m(b,c)).$$

In terms of maps:

$$m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m),$$

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mathrm{id}_G \times m} & G \times G \\ \downarrow{\scriptstyle m \times \mathrm{id}_G} & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{\quad m \quad} & G \end{array}$$

Algebra

- We did pay some price, but now the notion can be easily generalized to any category with tensor product generalization of Cartesian product of sets.
- Semi-group in category of sets
  ▶ Algebra in category of vector spaces.
- Set G ▶ Vector space A.
- Map $m : G \times G \to G$ ▶ Linear map $m : A \otimes A \to A$.
- Associativity is the same diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes m} & A \otimes A \\ \Big\downarrow{\scriptstyle m \otimes \mathrm{id}_A} & & \Big\downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array}$$

Algebra in tensor category

- We did pay some price, but now the notion can be easily generalized to any category with tensor product generalization of Cartesian product of sets.
- Semi-group in category of sets
  ▶ Algebra in any tensor category.
- Set G ▶ Object A.
- Map $m : G \times G \to G$ ▶ Morphism $m : A \otimes A \to A$.
- Associativity is the same diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes m} & A \otimes A \\ \Big\downarrow{\scriptstyle m \otimes \mathrm{id}_A} & & \Big\downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{\quad m \quad} & A \end{array}$$

Pick basis

- To express linear maps in terms of numbers, we have to choose a basis.
- An element in set G is a map from $\{\bullet\}$ to G.
- A vector $|\alpha\rangle$ and a dual vector $\langle\alpha|$ can both be translated into linear maps q_α and p^α:

$$q_\alpha : \mathbb{C} \to A, \qquad\qquad p^\alpha = (q_\alpha)^\dagger : A \to \mathbb{C},$$
$$\lambda \mapsto q_\alpha(\lambda) = \lambda|\alpha\rangle. \qquad\qquad |\psi\rangle \mapsto p^\alpha(|\psi\rangle) = \langle\alpha|\psi\rangle.$$

- An orthonormal basis is a set of vectors

$$\langle\alpha|\beta\rangle = \delta_{\alpha\beta}, \quad \sum_\alpha |\alpha\rangle\langle\alpha| = \mathrm{id}_A.$$

In terms of linear maps q_α, p^α

$$p^\alpha q_\beta = \delta_{\alpha\beta}\mathrm{id}_\mathbb{C}, \quad \sum_\alpha q_\alpha p^\alpha = \mathrm{id}_A.$$

Pick basis

- A unitary fusion category is tensor category with good structures such that one can pick a finite basis colored by topological charges/simple objects for any object.
- Now, given a composite anyon

$$A = \oplus i^{\oplus N_i^A},$$

where i labels the simple anyons, a basis of A is a set of morphisms ("linear maps" in generic tensor category) $q_{i,\alpha}^A : i \to A$, $p_A^{i,\alpha} = (q_{i,\alpha}^A)^\dagger : A \to i$, $\alpha = 1, \ldots, N_i^A$

$$p_A^{i,\alpha} q_{j,\beta}^A = \delta_{ij}\delta_{\alpha\beta}\mathrm{id}_i, \quad \sum_i \sum_{\alpha=1}^{N_i^A} q_{i,\alpha}^A p_A^{i,\alpha} = \mathrm{id}_A.$$

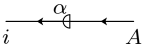
Pick basis

It is intuitive to use the following graphs for the basis:

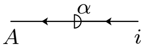
$= p^{i,\alpha}_{A},$

$= q^{A}_{i,\alpha}.$

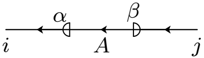
Our choice of symbol is to emphasize the similarity to the basis of Hilbert spaces (rotate the graph by 90° anticlockwise). They satisfy similar orthonormal and complete conditions:



$\sim \langle i, \alpha |,$

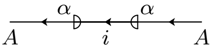


$\sim | \alpha, i \rangle,$



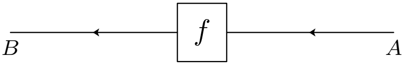
$\sim \langle i, \alpha | \beta, j \rangle = \delta_{ij}\delta_{\alpha\beta},$

$\sum_{i\alpha}$



$\sim \sum_{i\alpha} | \alpha, i \rangle \langle i, \alpha | = \mathrm{id}_A.$

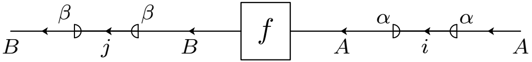
Any morphism $f : A \to B$



becomes a block diagonal matrix if we choose bases for A and B.
Insert identity,

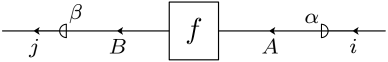
$$f = \mathrm{id}_B f \mathrm{id}_A = \sum_{j\beta, i\alpha} q^B_{j,\beta} p^{j,\beta}_B f q^A_{i,\alpha} p^{i,\alpha}_A$$

$$= \sum_{j\beta, i\alpha}$$

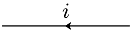


Express morphism by matrix or tensor

The i to j part reduces to a number

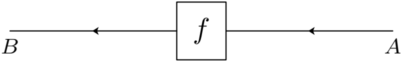


$$= f_{i,\beta\alpha}\delta_{ij}$$

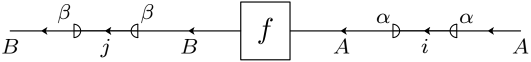


Thus

$$f =$$

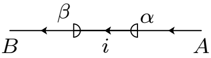


$$= \sum_{j\beta,i\alpha}$$



$$= \sum_{j\beta,i\alpha} q^B_{j,\beta} p^{j,\beta}_B f q^A_{i,\alpha} p^{i,\alpha}_A = \sum_{j\beta,i\alpha} q^B_{j,\beta} f_{i,\beta\alpha}\delta_{ij}\mathrm{id}_i p^{i,\alpha}_A$$

$$= \sum_{i\beta\alpha} f_{i,\beta\alpha} q^B_{i,\beta} p^{i,\alpha}_A = \sum_{i,\beta\alpha} f_{i,\beta\alpha}$$



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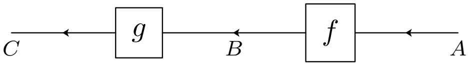
Tensor network with extra leg "color"

The composition of morphism is just matrix multiplication

$gf =$

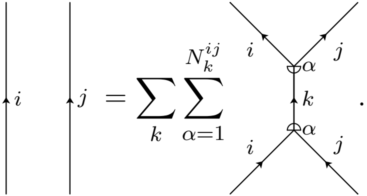


$$(gf)_{i,\beta\alpha} = \sum_{\gamma} g_{i,\beta\gamma} f_{i,\gamma\alpha}.$$

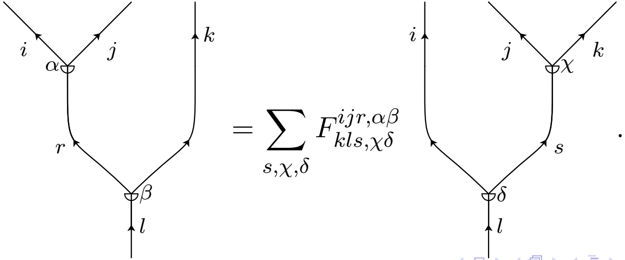
- The simple object i is the "color", the composite object $A = \oplus i^{\oplus N_i^A}$, or the number N_i^A, gives the bond dimension colored by i.
- When there is only a trivial color, these graphs reduce to usual tensor network.
- Usual tensor network can be used to build nontrivial bosonic states. These generalized tensor network in (braided) fusion category may be used to build anyonic states (in particular anyon condensate).

Fusion and braiding of the legs are nontrivial

Fix a basis for the tensor product $i \otimes j = \oplus k^{\oplus N_k^{ij}}$,



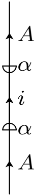
F-matrix (F-tensor or 6j-symbol) is just the matrix of associator $(i \otimes j) \otimes k \to i \otimes (j \otimes k)$ in this basis,



Structure coefficients of algebra

For an algebra A in a unitary fusion category, there is a multiplication morphism $m : A \otimes A \to A$. First take a basis of A,

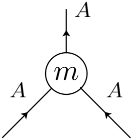
$\mathrm{id}_A =$ 

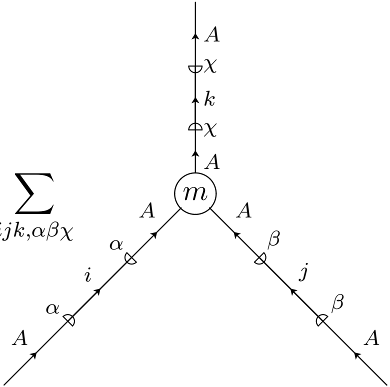
$= \sum_i \sum_{\alpha=1}^{N_i^A}$ 

$= \sum_i \sum_{\alpha=1}^{N_i^A} q_{i,\alpha}^{A} p_{A}^{i,\alpha}.$

Structure coefficients of algebra

The multiplication morphism m is then

$m =$


$= \sum_{ijk,\alpha\beta\chi}$


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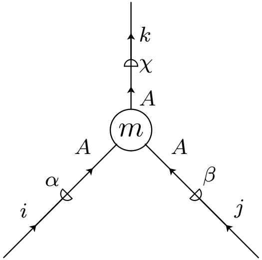
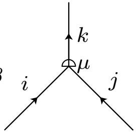
Structure coefficients of algebra

The central part can be expressed in terms of basis vertices $p_{ij}^{k,\mu}$



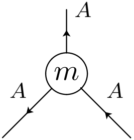
$$= \sum_{u} M_{i\alpha,j\beta}^{k\chi,\mu}$$



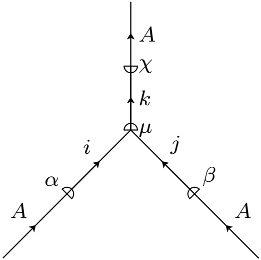
$$= \sum_{\mu} M_{i\alpha,j\beta}^{k\chi,\mu} p_{ij}^{k,\mu}.$$

Thus

$$m =$$



$$= \sum_{ijk,\alpha\beta\chi,\mu} M^{k\chi,\mu}_{i\alpha,j\beta}$$

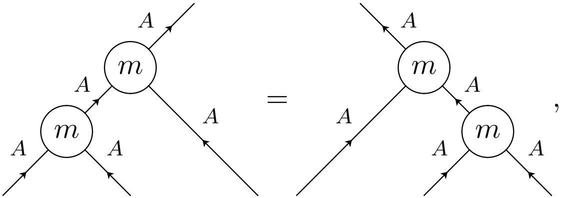


$$= \sum_{ijk,\alpha\beta\chi,\mu} M^{k\chi,\mu}_{i\alpha,j\beta} q^{A}_{k,\chi} p^{k,\mu}_{ij} \left(p^{i,\alpha}_{A} \otimes p^{j,\beta}_{A} \right).$$

$M^{k\chi,\mu}_{i\alpha,j\beta}$ is the "structure coefficients" of the algebra.

Conditions for the condensable algebra

Physically, we can use $m : A \otimes A \to A$ tensor to build a anyonic partition funcion $\mathcal{Z}_m$. The topoloical/retriangulation invariance of $\mathcal{Z}_m$ requires some self-consistency conditions of m, for example associativity,



Conditions for the condensable algebra

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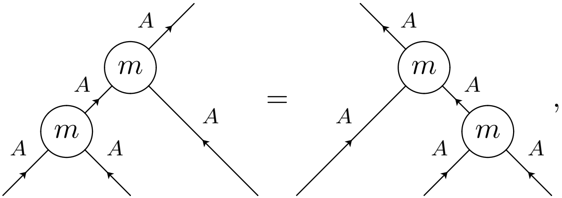
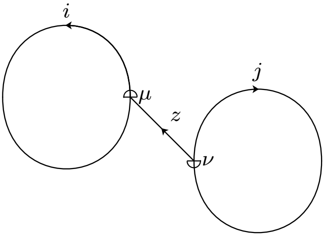


in terms of F-matrix and structure coefficients:

$$\sum_{\omega} M^{r\omega,\mu}_{i\alpha,j\beta} M^{l\delta,\nu}_{r\omega,k\chi} = \sum_{s\xi\psi\zeta} F^{ijr,\mu\nu}_{kls,\xi\psi} M^{s\zeta,\xi}_{j\beta,k\chi} M^{l\delta,\psi}_{i\alpha,s\zeta}.$$

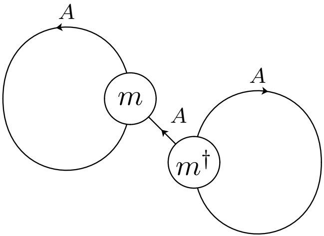
Groundstate overlap

A basis of groundstate subspace of a topological order described by UMTC $\mathcal{C}$ on genus 2 surface is



. (1)

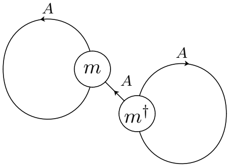
Assume there is a Lagrangian algebra A in $\mathcal{C}$, condensing A leads to a trivial topological order. Its unique ground state (up to a total normalization factor) on genus 2 surface is



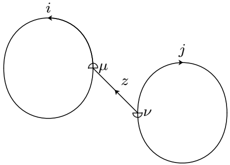
. (2)

Express (2) in terms of the basis (1).

Groundstate overlap



$$= \sum_{ijz\mu\nu} \sum_{\alpha\beta\chi} M^{i\alpha,\mu}_{i\alpha,z\chi} \left(M^{j\beta,\nu}_{z\chi,j\beta} \right)^*$$



$\sum_{\alpha\beta\chi} M^{i\alpha,\mu}_{i\alpha,z\chi} \left(M^{j\beta,\nu}_{z\chi,j\beta} \right)^*$ (up to a normalization factor) is the overlap between groundstates before and after anyon condensation.

TL, Xueda Wen, Liang Kong, and Xiao-Gang Wen, arXiv:1911.08470.