Matrix Formulation for Non-abelian Family

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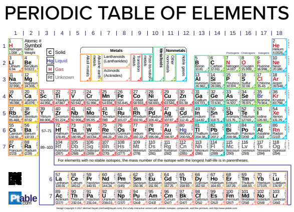
T. Lan, X.-G. Wen, PRL 119, 040403 (2017), arXiv:1701.07820
T. Lan, PhD Thesis Section 5.2, arXiv:1801.01210
T. Lan, PRB 100, 241102(R) (2019), arXiv:1908.02599

Outline

Non-abelian Family

Defined in [T. Lan, X.-G. Wen, PRL 119, 040403 (2017), arXiv:1701.07820], in analogy to chemical families:



- Abelian topological orders (TO) belong to the trivial family.
- TOs differing by abelian ones to be defined by reversible generalized hierarchy construction belong to the same non-abelian family.
- TOs in the same non-abelian family share similar non-abelian properties.

- Noble gas elements belong to the trivial chemically inert family.
- Elements differing by noble gas cores belong to the same chemical family.
- Elements in the same chemical family have similar chemical properties.

Abelian Topological Order and K-matrix Formulation

- An abelian topological order, or pointed modular tensor category, or metric group (G, q), can be described by a symmetric invertible integer matrix K whose diagonal entries are all even.
- Anyons are represented by integer vectors $\boldsymbol{l}$, up to the equivalence relation $\boldsymbol{l} \sim \boldsymbol{l} + K\boldsymbol{k}$, where $\boldsymbol{k}$ is an arbitrary integer vector.
- Fusion is by addition $\boldsymbol{l}_1 + \boldsymbol{l}_2$ then imposing the equivalence relation.
- Topological spin $s_{\boldsymbol{l}} = \frac{1}{2}\boldsymbol{l}^T K^{-1} \boldsymbol{l}$.

X. G. Wen and A. Zee, Phys. Rev. B 46, 2290 (1992)

Relations between the K-matrix Formulation and Others

- Consider a free abelian group $\mathbb{Z}^\kappa$, where κ is the rank of K.
  K gives a non-degenerate quadratic form $q_K([\boldsymbol{l}]) = \exp(\mathrm{i}\pi \boldsymbol{l}^T K^{-1} \boldsymbol{l})$ on $\mathbb{Z}^\kappa / K\mathbb{Z}^\kappa$.
  Then $(\mathbb{Z}^\kappa / K\mathbb{Z}^\kappa, q_K)$ is a metric group.
- Chern-Simons theory:

$$\mathcal{L} = \frac{K_{IJ}}{4\pi} \varepsilon^{\mu\nu\lambda} a^I_\mu \partial_\nu a^J_\lambda.$$

  Effective ground state wave function multilayer Laughlin, polynomial part:

$$\prod (z_a^{(I)} - z_b^{(J)})^{K_{IJ}}.$$

Matrix Formulation For Non-abelian Families

- The difference between two TOs in the same non-abelian family can be encoded in a series of abelian anyons $\boldsymbol{a} = (a_1, \ldots, a_\kappa)^T$, together with a κ-dimensional symmetric invertible matrix K. Entries of K are integers minus mutual statistics.
- Conversely, pick a root $\mathcal{C}$ (a TO with the smallest rank) in a family, all other TOs in the same family can be efficiently generated and represented by $(\mathcal{C}, \boldsymbol{a}, K)$.
- Generalizing the K matrix formulation of abelian TOs.

Generalized Hierarchy Construction – One-step

1. Starting from topological order $\mathcal{C}$, $(N^{ij}_k, s_i, c^{\mathcal{C}}, S^{\mathcal{C}}, T^{\mathcal{C}}, \ldots)$.
2. Choose an abelian anyon a_c in $\mathcal{C}$ and an even integer m_c.
   Also need to compute the mutual statistics t_{i,a_c} between $i \in \mathcal{C}$ and a_c, $d_i \mathrm{e}^{2\pi \mathrm{i} t_{i,a_c}}/D = S^{\mathcal{C}}_{i\bar{a}_c}$. Let a_c condense into a Laughlin state $\Psi = \prod (z_a - z_b)^{M_c}$, $M_c = m_c - t_{a_c,a_c} = m_c - 2s_{a_c}$, and we obtain topological order $\mathcal{C}_{a_c,M_c}$.
3. Anyons in $\mathcal{C}_{a_c,M_c}$ are represented by (i, M), where $i \in \mathcal{C}$ and $M + t_{i,a_c}$ is an integer, up to the equivalence relation

$$(i, M) \sim (i \otimes a_c, M + M_c).$$

   Anyon types in $\mathcal{C}_{a_c,M_c}$ correspond to equivalence classes.
4. Fusion rules and spin

$$(i, M) \otimes (j, L) = \bigoplus_k N^{ij}_k (k, M + L), \quad s_{(i,M)} = s_i + \frac{M^2}{2M_c}.$$

   Then impose the above equivalence relation.

Effective Wavefunction of Anyons

- Let $|\{\xi_a, z_a\}, \mu, M^2\rangle$ be a quantum state on a manifold M^2 with anyons of charge ξ_a at positions z_a. When the topology of M^2 is nontrivial or there are non-abelian anyons, the charges and positions are not enough to fix a state topological degeneracy, thus we need some additional label μ.
- By an effective anyon wavefunction $\Psi(\{\xi_a, z_a\}, \mu, M^2)$, we mean the following state certain superpostion of anyon states

$$\sum_{\xi_a, z_a, \mu} \Psi(\{\xi_a, z_a\}, \mu, M^2)|\{\xi_a, z_a\}, \mu, M^2\rangle,$$

which may be potentially in some different quantum phase.
- For simplicity, we consider only abelian anyons on sphere S^2 (or infinite plane $\mathbb{R}^2$), and drop μ, M^2.

Anyon Condensation

For example, anyon condensation corresponds to the effective wavefunction

$$\Psi(\{\xi_a, z_a\}) = \begin{cases} 1, & \xi_a \text{ all condensed,} \\ 0, & \text{some } \xi_a \text{ not condensed.} \end{cases}$$

Namely, for the condensed anyons, all the positions are equally possible; they are in a zero-total-momentum state. We may also say condensed anyons form a trivial state.

The possible form of effective wavefunction heavily depends on the statistics of the anyons. The above one requires that all condensed anyons are bosons. Thus better named boson (bosonic anyon) condensation

Hierarchy Construction

Original Idea due to Haldane and Halperin in quantum hall states. But we want to generalize it in any (non-abelian) topological orders.

Let abelian anyon a_c condense into a Laughlin state

$$\Psi(\{\xi_a = a_c, z_a\}) = \prod_{a<b} (z_a - z_b)^{M_c}.$$

M_c must be an even integer if a_c is a boson, or an odd integer if a_c is a fermion. Anyons?

Hierarchy Construction

Let abelian anyon a_c condense into a Laughlin state

$$\Psi(\{\xi_a = a_c, z_a\}) = \prod_{a<b} (z_a - z_b)^{M_c}.$$

Consider exchanging two a_c anyons, we obtain:

- Phase factor $\mathrm{e}^{2\pi \mathrm{i} \frac{M_c}{2}}$ from the wave function;
- Phase factor $\mathrm{e}^{2\pi \mathrm{i} s_{a_c}}$ from anyonic statistics.
- To be consistent, total phase factor must be 1:

$$\frac{M_c}{2} + s_{a_c} \in \mathbb{Z}.$$

So we need to take $M_c = m_c - 2s_{a_c}$, where m_c is an even integer.

Anyons in the new state

Anyon i can be non-abelian in the old state may be dressed with a flux M in the new state.

$$\Psi_{(i,M)}(\{\xi' = i, z', \xi_a = a_c, z_a\}) = \prod_a (z' - z_a)^M \prod_{a<b} (z_a - z_b)^{M_c}.$$

Thus an anyon in the new state is represented by a pair (i, M).
Again, M can not be arbitrary. If a_c has trivial mutual statistics with i, M can be any integer; otherwise, consider moving a_c around (i, M) and we obtain:

- Phase factor $e^{2\pi i M}$ from the flux M;
- Phase factor $e^{2\pi i t_{i,a_c}}$ from the mutual statistics between a_c and i.
  $e^{2\pi i t_{i,a_c}} = D S_{i\bar{a}_c}/d_i$, $t_{a_c,a_c} = 2 s_{a_c}$
- To be consistent, total phase factor must be 1:

$$M + t_{i,a_c} \in \mathbb{Z}.$$

Anyons in the new state

$$\Psi_{(i,M)}(\{\xi' = i, z', \xi_a = a_c, z_a\}) = \prod_a (z' - z_a)^M \prod_{a<b} (z_a - z_b)^{M_c}.$$

The spin of (i, M) is given by the spin of i plus the “spin” of the the flux M:

$$s_{(i,M)} = s_i + \frac{M^2}{2M_c}.$$

To fuse anyons $(i, M), (j, L)$ in the new state, just fuse i, j as in the old state, and add up the flux:

$$(i, M) \otimes (j, L) = \bigoplus_k N_k^{ij} (k, M + L).$$

But note that this is not the final fusion rules.

Anyons in the new state

The anyon a_c dressed with a flux M_c is a "trivial excitation" in the new state:

$$\Psi_{(a_c,M_c)} = \prod_{a}^{n}(z' - z_a)^{M_c} \prod_{a<b}^{n}(z_a - z_b)^{M_c} = \prod_{a<b}^{n+1}(z_a - z_b)^{M_c}.$$

$$(a_c, M_c) \sim (\mathbf{1}, 0).$$

Therefore, anyons (i, M) in the new state are subject to the equivalence relation

$$(i, M) \sim (i \otimes a_c, M + M_c).$$

After imposing the equivalence relation one obtains the final fusion rules in the new state.

Generalized Hierarchy Construction

- Valid construction at categorical level, for any braided fusion categories.
- Reversible. (In $\mathcal{C}_{a_c,M_c}$ choose the unit flux $(\mathbf{1},1)$ as a'_c and $M'_c = -1/M_c$.) Defines a valid equivalence relation between topological orders.

$$\mathcal{C} \xleftrightarrow[\text{generalized hierarchy construction}]{\sim} \mathcal{D}$$

Equivalence class
= Orbit of generalized hierarchy construction
= Non-abelian Family

Generalized Hierarchy Construction

- The rank is $N^{\mathcal{C}_{a_c,M_c}} = |M_c|N^{\mathcal{C}}$, $M_c = m_c - t_{a_c,a_c} = m_c - 2s_{a_c}$, with m_c even. If a_c is not a self boson/fermion $2s_{a_c} \neq 0, 1 \mod 2$, can choose proper m_c s.t. $-1 < M_c < 1$, to reduce the rank.
- If abelian bosons or fermion have non-trivial mutual statistics among them, can also reduce the rank by 2 steps of generalized hierarchy construction.

Root topological orders

(1) have the smallest rank among a non-abelian family;
or equivalently
(2) Abelian anyons in roots are all bosons or fermions with trivial mutual statistics among them.
The subcategory of abelian anyons $\mathcal{C}_{pt}$ is symmetric, namely the representation category of an abelian (super-)group.

Common Properties of a Non-abelian Family

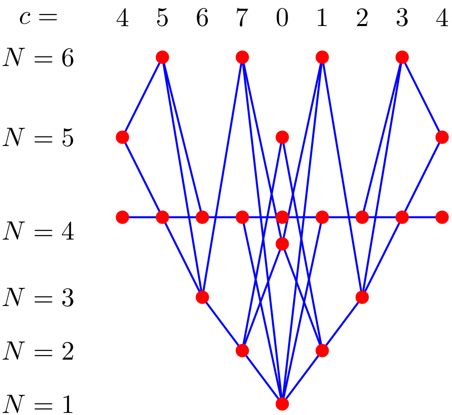
- Quantum dimensions, $d_{(i,M)} = d_i$.
- $c^{\mathcal{C}_{a_c,M_c}} = c^{\mathcal{C}} + |M_c|/M_c$.
  Fractional part of the central charge, $c \mod 1$.
- If mutual statistics between i and a_c is trivial, $t_i = 0$, then the self/mutual statistics of $(i, M = 0)$ is the same as i.
  The subset of anyons that have trivial mutual statistics with all abelian anyons. The Müger centralizer of abelian anyons $(\mathcal{C}_{pt})'_{\mathcal{C}}$
- …

Non-abelian Families

abelian family

N_c	D^2	$s_1, s_2, \cdots$	$d_1, d_2, \cdots$
1_0	1	0	1
2_1	2	$0, \frac{1}{4}$	
2_7	2	$0, \frac{3}{4}$	
3_2	3	$0, \frac{1}{3}, \frac{1}{3}$	
3_6	3	$0, \frac{2}{3}, \frac{2}{3}$	
4_0	4	$0, 0, 0, \frac{1}{2}$	
4_0	4	$0, 0, \frac{1}{4}, \frac{3}{4}$	
4_1	4	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$	
4_7	4	$0, \frac{7}{8}, \frac{7}{8}, \frac{1}{2}$	
4_2	4	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$	
4_6	4	$0, \frac{3}{4}, \frac{3}{4}, \frac{1}{2}$	
4_3	4	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$	
4_5	4	$0, \frac{5}{8}, \frac{5}{8}, \frac{1}{2}$	
4_4	4	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	
5_0	5	$0, \frac{1}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5}$	
5_4	5	$0, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}$	
6_1	6	$0, \frac{1}{12}, \frac{1}{12}, \frac{3}{4}, \frac{1}{3}, \frac{1}{3}$	
6_7	6	$0, \frac{11}{12}, \frac{11}{12}, \frac{1}{4}, \frac{2}{3}, \frac{2}{3}$	
6_3	6	$0, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{7}{12}, \frac{7}{12}$	
6_5	6	$0, \frac{3}{4}, \frac{2}{3}, \frac{2}{3}, \frac{5}{12}, \frac{5}{12}$	

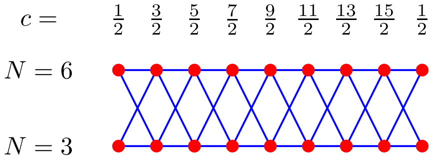
If the root contains only one abelian anyon (trivial one), the corresponding non-abelian family is the stacking of the root with the abelian family.



Non-abelian Families

$SU(2)_2$ Ising non-abelian family

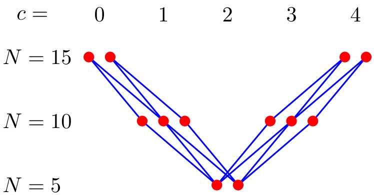
N_c	D^2	$s_1, s_2, \cdots$ $d_1, d_2, \cdots$
$3_{\frac{1}{2}}$	4	$0, \frac{1}{2}, \frac{1}{16}$ $1, 1, \sqrt{2}$
$3_{\frac{15}{2}}$	4	$0, \frac{1}{2}, \frac{15}{16}$
$3_{\frac{3}{2}}$	4	$0, \frac{1}{2}, \frac{3}{16}$ $SU(2)_2$
$3_{\frac{13}{2}}$	4	$0, \frac{1}{2}, \frac{13}{16}$
$3_{\frac{5}{2}}$	4	$0, \frac{1}{2}, \frac{5}{16}$
$3_{\frac{11}{2}}$	4	$0, \frac{1}{2}, \frac{11}{16}$
$3_{\frac{7}{2}}$	4	$0, \frac{1}{2}, \frac{7}{16}$
$3_{\frac{9}{2}}$	4	$0, \frac{1}{2}, \frac{9}{16}$
$6_{\frac{1}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{15}{16}, \frac{3}{16}$
$6_{\frac{15}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{16}, \frac{13}{16}$
$6_{\frac{3}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{16}, \frac{5}{16}$
$6_{\frac{13}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{15}{16}, \frac{11}{16}$
$6_{\frac{5}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{3}{16}, \frac{7}{16}$
$6_{\frac{11}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{13}{16}, \frac{9}{16}$
$6_{\frac{7}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{5}{16}, \frac{9}{16}$
$6_{\frac{9}{2}}$	8	$0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{11}{16}, \frac{7}{16}$



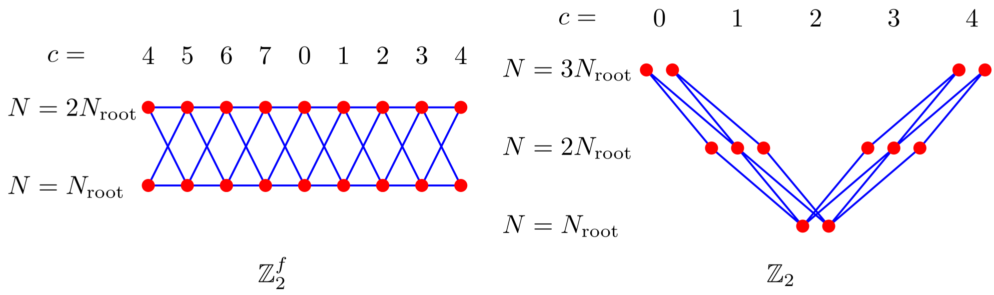
Non-abelian Families

$SU(2)_4$ non-abelian family

N_c	D^2	$s_1, s_2, \cdots$		$d_1, d_2, \cdots$
5_2	12	$0, 0, \frac{1}{8}, \frac{5}{8}, \frac{1}{3}$	$SU(2)_4$	$1, 1, \sqrt{3}, \sqrt{3}, 2$
5_2	12	$0, 0, \frac{7}{8}, \frac{3}{8}, \frac{1}{3}$		
10_1	24	$0, 0, \frac{3}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{3}{8}, \frac{5}{8}, \frac{1}{12}, \frac{1}{3}$		
10_1	24	$0, 0, \frac{3}{4}, \frac{3}{4}, \frac{1}{16}, \frac{1}{16}, \frac{9}{16}, \frac{9}{16}, \frac{1}{12}, \frac{1}{3}$	$\frac{U(1)_3}{\mathbb{Z}_2}$	
10_1	24	$0, 0, \frac{3}{4}, \frac{3}{4}, \frac{13}{16}, \frac{13}{16}, \frac{5}{16}, \frac{5}{16}, \frac{1}{12}, \frac{1}{3}$		
10_3	24	$0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{7}{8}, \frac{3}{8}, \frac{5}{8}, \frac{1}{3}, \frac{7}{12}$		
10_3	24	$0, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{16}, \frac{3}{16}, \frac{11}{16}, \frac{11}{16}, \frac{1}{3}, \frac{7}{12}$		
10_3	24	$0, 0, \frac{1}{4}, \frac{1}{4}, \frac{15}{16}, \frac{15}{16}, \frac{7}{16}, \frac{7}{16}, \frac{1}{3}, \frac{7}{12}$		
15_0	36	$0, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{24}, \frac{1}{24}, \frac{7}{8}, \frac{3}{8}, \frac{13}{24}, \frac{13}{24}, 0, 0, \frac{1}{3}$		
15_0	36	$0, 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{8}, \frac{19}{24}, \frac{19}{24}, \frac{7}{24}, \frac{7}{24}, \frac{5}{8}, 0, 0, \frac{1}{3}$		
15_4	36	$0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{23}{24}, \frac{23}{24}, \frac{1}{8}, \frac{5}{8}, \frac{11}{24}, \frac{11}{24}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$		
15_4	36	$0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{7}{8}, \frac{5}{24}, \frac{5}{24}, \frac{17}{24}, \frac{17}{24}, \frac{3}{8}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$		



Non-abelian Families



The pattern of a non-abelian family is determined by the abelian anyons in the root:

- Only the trivial abelian anyon,
  unique root, stacking with abelian family.
  $\Rightarrow$ root + K-matrix efficiently describes such family.
- $\mathbb{Z}_2^f$ fermion, 8-fold way, as Ising family
- $\mathbb{Z}_2$ boson, 2 roots, similar pattern as $SU(2)_4$ family.

Abelian (super-)group determines the tree pattern.

Hierarchy Construction in K-matrix formulation

In particular, all abelian topological orders belong to the same family, whose root has to be the trivial one. In this case, if $\mathcal{C}$ is described by $K^{\mathcal{C}}$, choose $a_c = \boldsymbol{l}_c$ and even integer m_c, $\mathcal{C}_{\boldsymbol{l}_c, m_c - \boldsymbol{l}_c^T K^{-1} \boldsymbol{l}_c}$ is the abelian topological order described by

$$K^{\mathcal{C}_{\boldsymbol{l}_c, m_c - \boldsymbol{l}_c^T K^{-1} \boldsymbol{l}_c}} = \begin{pmatrix} K^{\mathcal{C}} & \boldsymbol{l}_c \\ \boldsymbol{l}_c^T & m_c \end{pmatrix}.$$

K-matrix records the information how an abelian topological order is constructed. say, from the trivial root, an empty 0×0 K-matrix

Multiple-step Construction

- Now consider starting from a topological order $\mathcal{C}$ and performing one-step construction κ times.
- We need to compute the mutual statistics t_i at every step which is very involved.
- Instead we can record the integers = flux M + mutual statistics t_{i,a_c}, and forget the mutual statistics in the intermediate steps.

Multiple-step Construction

- The first step we take $a_1 \in \mathcal{C}_{pt}$ and even integer k_{11}.
- The second step we take an abelian anyon $(a_2 \in \mathcal{C}_{pt}, k_{21})$ and even integer k_{22}, where k_{21} is an integer.
- The third step we take an abelian anyon $((a_3 \in \mathcal{C}_{pt}, k_{31}), k_{32})$ and even integer k_{33}, where k_{31}, k_{32} are integers.
- Keep moving on and we see that the steps can be summarized by a_I and k_{IJ}:

$$\begin{matrix} a_1 & k_{11} & & \\ a_2 & k_{21} & k_{22} & \\ a_3 & k_{31} & k_{32} & k_{33} \end{matrix}$$

- Then we just use the mutual statistics t_{a_I,a_J} in $\mathcal{C}$ to build a symmetric matrix $K_{IJ} = k_{IJ} - t_{a_I,a_J}$. One-step $M_c = m_c - t_{a_c,a_c}$

Multiple-step Construction

This K matrix can be justified by the following multilayer effective Laughlin wavefunction:

$$\Psi(\{\xi_a^{(I)} = a_I, z_a^{(I)}\}) = \prod (z_a^{(I)} - z_b^{(J)})^{K_{IJ}},$$

where $I = 1, \ldots, \kappa$ labels the layer.

Denote by $t_{i,a}$ the mutual statistics $e^{2\pi i t_{i,a}}$ is the phase factor of braiding a around i between anyon i and abelian anyon a in $\mathcal{C}$. By a similar argument as in the one-step case, we know that

- $K_{IJ} + t_{a_I,a_J} \in \mathbb{Z}$,
- $K_{II} + t_{a_I,a_I} = K_{II} + 2s_{a_I} \in 2\mathbb{Z}$.

If $\mathcal{C}$ is a root, K is an integer matrix and K_{II} is even when a_I is a boson and odd when a_I is a fermion.

Multiple-step Construction

The new anyons are now labeled by $(i, \boldsymbol{L})$, where $\boldsymbol{L}$ is a κ-dimensional vector describing the flux from each layer:

$$\Psi_{(i,\boldsymbol{L})}(\{\xi' = i, z', \xi_a^{(I)} = a_I, z_a^{(I)}\}) = \prod (z' - z_a^{(I)})^{\boldsymbol{L}_I} \prod (z_a^{(I)} - z_b^{(J)})^{K_{IJ}}.$$

Again $\boldsymbol{L}$ is constrained by mutual statistics:

$$\boldsymbol{L}_I + t_{i,a_I} \in \mathbb{Z}.$$

Multiple-step Construction

We can formally denote by $\boldsymbol{a} = (a_I)$, and the new state by $\mathcal{C}_{\boldsymbol{a},K}$:

- Equivalence relation $(i, \boldsymbol{L}) \sim (i \otimes a_I, \boldsymbol{L} + K_I)$, where K_I is the Ith column vector of K. Or, for any integer vector $\boldsymbol{k}$ $(i, \boldsymbol{L}) \sim (i \otimes \boldsymbol{k}^T \boldsymbol{a}, \boldsymbol{L} + K\boldsymbol{k})$.
- Fusion is $(i, \boldsymbol{L}) \otimes (j, \boldsymbol{M}) = \oplus_k N_k^{ij} (k, \boldsymbol{L} + \boldsymbol{M})$.
- The spin of $(i, \boldsymbol{L})$ is $s_{(i,\boldsymbol{L})} = s_i + \frac{1}{2} \boldsymbol{L}^T K^{-1} \boldsymbol{L}$.
- The S matrix is

$$S_{(i,\boldsymbol{L})(j,\boldsymbol{M})} = \frac{1}{\sqrt{|\det K|}} S_{ij} \mathrm{e}^{-2\pi \mathrm{i} \boldsymbol{L}^T K^{-1} \boldsymbol{M}}.$$

- The rank is $N^{\mathcal{C}_{\boldsymbol{a},K}} = |\det K| N^{\mathcal{C}}$, and the chiral central charge changes by the index of K, $c^{\mathcal{C}_{\boldsymbol{a},K}} = c^{\mathcal{C}} + \mathrm{sgn} K$.
- The above is the same as κ times of one-step constructions, which can be proven by induction.

The Basis Independent Categorical Formulation

T. Lan, PRB 100, 241102(R) (2019), arXiv:1908.02599

Notation fixing:

- $\mathcal{C}$: a braided fusion category.
- $\alpha_{A,B,C}, c_{A,B}$: associator and braiding in $\mathcal{C}$.
- $\underline{\mathcal{C}_{pt}}$: the abelian group corresponding to the pointed subcategory $\mathcal{C}_{pt}$.
- $t : \mathrm{Irr}(\mathcal{C}) \times \underline{\mathcal{C}_{pt}} \to \mathbb{Q}$: the mutual statistics between simple objects and pointed ones, namely $\mathrm{e}^{2\pi \mathrm{i} t(i,a)} = \frac{1}{d_i} \mathrm{Tr}\, c_{a,i} c_{i,a}$.
- $\mathbb{Z}^\kappa$: free abelian group with κ generators. It can be naturally extended to a κ dimensional vector space over $\mathbb{Q}$.
- $x, y, \ldots$: elements in $\mathbb{Z}^\kappa$.
- $\overline{\mathbb{Z}}^\kappa := \mathrm{Hom}(\mathbb{Z}^\kappa, \mathbb{Q})$: the “dual space”, the space of $\mathbb{Q}$-linear functions.
- $f(-), g(-), \ldots$, or simply f, g: functions in $\overline{\mathbb{Z}}^\kappa$.

The Basis Independent Categorical Formulation

Let $K : \mathbb{Z}^\kappa \times \mathbb{Z}^\kappa \to \mathbb{Q}$ be a non-degenerate symmetric bilinear form. It defines an isomorphism from $\mathbb{Z}^\kappa$ to $\overline{\mathbb{Z}}^\kappa$, by

$$x \mapsto K(x, -) = K(-, x).$$

Denote the inverse map by $\tilde{K}$, thus

$$\tilde{K}(K(x, -)) = x, \quad K(\tilde{K}(f), x) = f(x).$$

There is then a natural non-degenerate symmetric bilinear form $\overline{K}$ on $\overline{\mathbb{Z}}^\kappa$ induced from K, via

$$\overline{K}(f, g) = K(\tilde{K}(f), \tilde{K}(g)) = f(\tilde{K}(g)) = g(\tilde{K}(f)).$$

If one chooses a basis of $\mathbb{Z}^\kappa$ and the corresponding dual basis of $\overline{\mathbb{Z}}^\kappa$, the matrix of K and $\overline{K}$ are inverse to each other.

The Basis Independent Categorical Formulation

We also need to choose κ abelian anyons for each step.
This is concluded in a group homomorphism $\boldsymbol{a} : \mathbb{Z}^\kappa \to \underline{\mathcal{C}_{pt}}$.
In view of the multilayer effective wavefunction, the addition of above groups means to put two layers together; a group automorphism of $\mathbb{Z}^\kappa$ corresponds to a recombination of layers.

The bilinear form K needs to satisfy the even integral condition, namely $\forall x, y$,

$$K(x,y) + t(\boldsymbol{a}(x), \boldsymbol{a}(y)) \in \mathbb{Z},$$

and

$$K(x,x) + t(\boldsymbol{a}(x), \boldsymbol{a}(x)) \in 2\mathbb{Z}.$$

Our final goal is to construct a "semi-direct product" of $\mathcal{C}$ and $\overline{\mathbb{Z}^\kappa}/K(\mathbb{Z}^\kappa, -)$.

The Basis Independent Categorical Formulation

For a κ step construction, first define an auxillary category $\mathcal{C}^{\uparrow}_{\boldsymbol{a},K}$:

- $\mathcal{C}^{\uparrow}_{\boldsymbol{a},K}$ is graded by $\overline{\mathbb{Z}}^{\kappa}/K(2\ker \boldsymbol{a}, -)$. not faithful

  Quotiented by $K(2\ker \boldsymbol{a}, -)$ to make things finite. $2\ker \boldsymbol{a}$ instead of $\ker \boldsymbol{a}$ is for a simpler expression of α, c.
- Take a representative $f \in \overline{\mathbb{Z}}^{\kappa}$ the flux, the component $(\mathcal{C}^{\uparrow}_{\boldsymbol{a},K})_f$ is a full subcategory of $\mathcal{C}$ with simple objects i satisfying $f(-) + t(i, \boldsymbol{a}(-)) \in \mathbb{Z}$ [$K(x,-)$ is an integer for $x \in \ker \boldsymbol{a}$, so well defined for $f + K(2\ker \boldsymbol{a}, -)$].
- Denote the simple objects in $\mathcal{C}^{\uparrow}_{\boldsymbol{a},K}$ by i_f. Define the tensor product and braiding in $\mathcal{C}^{\uparrow}_{\boldsymbol{a},K}$:

$$i_f \otimes j_g = (i \otimes j)_{f+g} = \oplus_k N^{ij}_k k_{f+g}, \tag{1}$$

$$\alpha_{i_f, j_g, k_h} = \alpha_{i,j,k}, \tag{2}$$

$$c_{i_f, j_g} = c_{i,j} \mathrm{e}^{\mathrm{i}\pi \overline{K}(f,g)}. \tag{3}$$

(3) is independent of the choice of representative. $K(2\ker \boldsymbol{a}, -)$ is the largest normal subgroup for this to be true. Thus $\mathcal{C}^{\uparrow}_{\boldsymbol{a},K}$ becomes a braided fusion category.

The Basis Independent Categorical Formulation

- Observe that for any $x \in \mathbb{Z}^\kappa$, $\boldsymbol{a}(x)_{K(x,-)} \in \mathcal{C}^{\uparrow}_{\boldsymbol{a},K}$ is a self boson and mutually trivial to any object i_f.
- In other words, $\{\boldsymbol{a}(x)_{K(x,-)},\ x \in \mathbb{Z}^\kappa\}$ generates a symmetric fusion subcategory in the Müger center of $\mathcal{C}^{\uparrow}_{\boldsymbol{a},K}$, which is braided equivalent to $\mathrm{Rep}(\mathbb{Z}^\kappa/2\ker\boldsymbol{a})$.
- Condense it (take the category of local modules over $\mathrm{Fun}(\mathbb{Z}^\kappa/2\ker\boldsymbol{a})$), and we obtain the final result

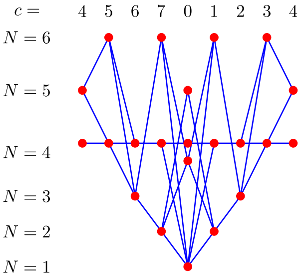
$$\mathcal{C}_{\boldsymbol{a},K} = (\mathcal{C}^{\uparrow}_{\boldsymbol{a},K})^{\mathrm{loc}}_{\mathrm{Fun}(\mathbb{Z}^\kappa/2\ker\boldsymbol{a})}.$$

  Roughly speaking, this imposes the equivalence relation $i_f \sim i_f \otimes \boldsymbol{a}(x)_{K(x,-)}$. The grading (flux) range reduces to

$$\frac{\overline{\mathbb{Z}}^\kappa/K(2\ker\boldsymbol{a},-)}{\mathbb{Z}^\kappa/2\ker\boldsymbol{a}} \cong \overline{\mathbb{Z}}^\kappa/K(\mathbb{Z}^\kappa,-).$$

- Since the condensed anyons $\boldsymbol{a}(x)_{K(x,-)}$ are abelian and in the Müger center, the fusion rules and braidings thus also T, S matrices are preserved. But the associator α (F-symbol) gets complicated.

Equivalence Relation of $\boldsymbol{a}, K$



Starting from the same topological order $\mathcal{C}$, different paths of construction may result in the same topological order. It is natural to ask what is the equivalence relation of $(\boldsymbol{a}, K)$. For now, we know three ways to generate equivalent $\mathcal{C}_{\boldsymbol{a},K}$:

- The equivalence between the starting point $F: \mathcal{C} \simeq \mathcal{D}$ naturally give rise to equivalence $\mathcal{C}_{\boldsymbol{a},K} \simeq \mathcal{D}_{F(\boldsymbol{a}),K}$.
- "Integer linear recombination" of a_I, $W \in GL(\kappa, \mathbb{Z})$ namely W is an integer matrix with $\det W = \pm 1$, or an automorphism of $\mathbb{Z}^\kappa$, $\mathcal{C}_{\boldsymbol{a},K} \simeq \mathcal{C}_{W\boldsymbol{a},WKW^T}$. We call such transformation as the $GL(\mathbb{Z})$ transformation.

Equivalence Relation of $\boldsymbol{a}, K$

- The reversibility of one-step construction means that the topological order constructed from $\mathcal{C}$ with $\begin{pmatrix} a_1 = a_c \\ a_2 = \mathbf{1} \end{pmatrix}$, $K = \begin{pmatrix} M_c & 1 \\ 1 & 0 \end{pmatrix}$ is equivalent to $\mathcal{C}$. Note that under $GL(\mathbb{Z})$ transformation, $\begin{pmatrix} M_c & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} -t_{a_c,a_c} & 1 \\ 1 & 0 \end{pmatrix}$. We have $(\boldsymbol{a}, K) \sim \left(\boldsymbol{a} \oplus \begin{pmatrix} b \\ \mathbf{1} \end{pmatrix}, K \oplus \begin{pmatrix} -t_{b,b} & 1 \\ 1 & 0 \end{pmatrix}\right)$ for any abelian anyon b. We refer to $\left(\begin{pmatrix} b \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} -t_{b,b} & 1 \\ 1 & 0 \end{pmatrix}\right)$ as the "trivial bilayer".

Equivalence Relation of $\boldsymbol{a}, K$

The complete equivalence relation is still an open question.

Conjecture

$\mathcal{C}_{\boldsymbol{a},K}$ and $\mathcal{C}_{\boldsymbol{a}',K'}$ (with exactly the same chiral central charge, not modulo 8) are equivalent if and only if, up to automorphisms of $\mathcal{C}$ and $GL(\mathbb{Z})$ transformations, $(\boldsymbol{a} \oplus \boldsymbol{b}, K \oplus X) \sim (\boldsymbol{a}' \oplus \boldsymbol{b}', K' \oplus X')$ where $(\boldsymbol{b}, X)$ and $(\boldsymbol{b}', X')$ are direct sums of trivial bilayers $\left(\begin{pmatrix} b \\ \mathbf{1} \end{pmatrix}, \begin{pmatrix} -t_{b,b} & 1 \\ 1 & 0 \end{pmatrix} \right)$.

Closely related:

- Find the canonical form of $\boldsymbol{a}, K$ under $GL(\mathbb{Z})$ transformation.
- What if the central charges differ by multiples of 8?
- Given $\mathcal{C}, \mathcal{D}$, how to determine if they are in the same family? If they are, what is $\boldsymbol{a}, K$ such that $\mathcal{D} = \mathcal{C}_{\boldsymbol{a},K}$?

Conclusion and Outlook

Given a topological order $\mathcal{C}$ preferably a root, any in the same family can be efficiently represented by $\mathcal{C}_{a,K}$.

Open questions:

- Relation to (anyonic) effective field theory?
- What corresponds to the "Period" of chemical elements, rows in the Periodic Table? Operation moving between non-abelian families?
- Classify root topological orders? still difficult
  = Classify all.
- Other constructions via different forms of effective wavefunction?
  Formula: Condense anyons $\{\xi_a\}$ into state $\Psi(\{\xi_a, z_a\})$.

Thanks for attention!