A Framework for 2+1D Topological Phases with Symmetries

Tian Lan

Institute for Quantum Computing
University of Waterloo

In collaboration with Liang Kong, Xiao-Gang Wen

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Topological Phases of Matter

- Quantum phases of matter ($H = \sum_i H_i, V = \otimes_i V_i$) with local structures and a finite energy gap.
- Symmetry $U_g H U_g^{-1} = H$, $g \in G_H$.
- Used to be considered solved by Landau symmetry breaking theory.
- Exotic phases with "topological" nature discovered.
- Fractional quantum Hall: fractional charges, fractional statistics, protected gapless edge states.
  $\to$ Intrinsic topological order, not requiring any symmetry

  X.-G. Wen, Int. J. Mod. Phys. B 4, 239 (1990); X.-G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990)
- Topological insulator: symmetry protected gapless (conducting) edge states.
  $\to$ Symmetry protected topological (SPT) phases, no intrinsic topological order

  X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G Wen, Phys. Rev. B 87, 155114 (2013), Science 338, 1604 (2012)

Topological Phases in Different Dimensions

	1+1D	2+1D	3+1D
Symmetry breaking	✓	✓	✓
SPT	✓	✓	✓
Topological order	×	✓	✓
???	×	×	✓

Complete classification in 1+1D

- Symmetry breaking phases $G \subset G_H$
  G_H is the symmetry group of the Hamiltonian
  G is the symmetry group of the ground states
- 1+1D Topological phases
  $G \subset G_H, \mathrm{pRep}(G)$ (or $H^2(G, U(1))$)
  Symmetry breaking, SPT

X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 83, 035107 (2011)
N. Schuch, D. Perez-Garcia, and I. Cirac, Phys. Rev. B 84, 165139 (2011)

But in 2+1D need to combine symmetry with topological order.

Framework

2+1D Topological Phases

$$G \subset G_H, \quad \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, \quad c$$

$G \subset G_H$ — Symmetry breaking

$\mathcal{E}, \mathcal{C}, \mathcal{M}$ — unitary braided fusion categories (UBFC)

fusion and braiding (statistics) of quasiparticles (anyon model)

- $\mathcal{E}$ local excitations carrying group representations
  symmetric fusion category, $\mathrm{Rep}(G)$ or $\mathrm{sRep}(G^f)$
- $\mathcal{C}$ $\mathcal{E}$ plus "anyons", all bulk excitations
  UBFC with Müger center $\mathcal{E}$, $\mathrm{UMTC}_{/\mathcal{E}}$
- $\mathcal{M}$ $\mathcal{C}$ plus "gauged symmetry defects", excitations in the "gauged" phase
  minimal modular extension of $\mathcal{C}$
  captures some information of "invertible" stack to trivial phases

c — central charge, to address E_8 states that are invisible to $\mathcal{M}$.

Local Excitations $\mathcal{E}$

- The “non-exotic” excitations
  that can be created by local operators.
- Carry symmetry charges/representations.
- Boson systems $\mathcal{E} = \mathrm{Rep}(G)$, fusion is tensor product of representations, braiding is all trivial.
- Fermion systems $\mathcal{E} = \mathrm{sRep}(G^f)$, whose fusion is the same as $\mathrm{Rep}(G)$, but braiding is different. The phase of braiding two fermions is changed to -1.
- $\mathcal{E}$ uniquely determines the symmetry group G.

Tannaka-Krein Duaility; P. Deligne, Catégories tensorielles, Mosc. Math. J. 2 (2002), no. 2, 227–248

2+1D Topological Phases

$$G \subset G_H, \quad \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, \quad c$$

All Bulk Excitations $\mathcal{C}$

- Local excitations $\mathcal{E}$ + "exotic" ones (fractional/non-Abelian anyons)
- Braiding non-degeneracy:
  "Exotic" excitations must be detectable remotely.
  The excitations with trivial mutual braiding statistics with all excitations in $\mathcal{C}$ must be the local ones.
  Müger center of $\mathcal{C}$ coincide with $\mathcal{E}$. UMTC over $\mathcal{E}$, $\mathrm{UMTC}_{/\mathcal{E}}$

  $\mathcal{C}$ has anomaly if not satisfied, requiring a 3+1D topological ordered bulk.
- Extreme case $\mathcal{E}=\mathcal{C}, c=0$: SPT
- SPT has non-trivial classification.
  Need more information than $\mathcal{E}\subset\mathcal{C}$.

2+1D Topological Phases

$$G\subset G_H,\quad \mathcal{E}\subset\mathcal{C}\subset\mathcal{M},\quad c$$

Gauging/Modular Extension $\mathcal{M}$

- Promote extrinsic symmetry defects to dynamical excitations, "gauge the symmetry"

  M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012)

  $\rightarrow$ bosonic topological order with no symmetry
- A larger anyon model (UMTC) that contains all bulk excitations $\mathcal{C}$ plus gauged symmetry defects
  $=$ minimal modular extension of $\mathcal{C}$
- The gauged symmetry defects can detect $\mathcal{E}$ via braidings.
- "Minimal" in the sense that gauged symmetry defects must have non-trivial mutual statistics with at least one local excitation in $\mathcal{E}$. The centralizer of $\mathcal{E}$ in $\mathcal{M}$ is $\mathcal{C}$.

2+1D Topological Phases

$$G \subset G_H, \quad \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, \quad c$$

Gauging/Modular Extension $\mathcal{M}$

- UMTC $\mathcal{M}$ captures most information and fixes the chiral central charge c modulo 8.
- Only ambiguity left is invertible states with no anyons and central charge in $8\mathbb{Z}$.
  They are generated by the E_8 state, fixed by the chiral central charge c.
- Minimal modules extension may not exist, in which case $\mathcal{C}$ has symmetry anomaly, requiring a 3+1D SPT bulk.

2+1D Topological Phases

$$G \subset G_H, \quad \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, \quad c$$

Examples

Assume no symmetry breaking in the following.

Toric code model with no symmetry ($\mathbb{Z}_2$ gauge theory)

$$\mathcal{E} = \{1\}, \mathcal{C} = \mathcal{M} = \mathcal{M}_{tc} = \{1, e, m, f\}, c = 0.$$

$\mathbb{Z}_2^f$ invertible fermionic topological orders

$$\mathcal{E} = \mathcal{C} = \mathrm{sRep}(\mathbb{Z}_2^f) = \{1, f\}.$$

$$f \otimes f = 1, s_f = 1/2$$

16-fold way 16 $\mathcal{M}$ with central charge $c = \frac{n}{2}$.

8 Ising type $\{1, f, \sigma\}, d_\sigma = \sqrt{2}$. $p + ip$ superconductors. σ "vortex", flux of gauged $\mathbb{Z}_2^f$.
8 Abelian: 4 $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion, 4 $\mathbb{Z}_4$ fusion. Integer quantum hall states.

Examples

$\mathbb{Z}_2$ SPT

$$\mathcal{E} = \mathcal{C} = \mathrm{Rep}(\mathbb{Z}_2) = \{1_+, 1_-\}, \quad 1_- \otimes 1_- = 1_+.$$

$$\mathcal{M} = \mathcal{M}_{tc} = \{1 \sim 1_+, e \sim 1_-, m, f\}.$$

$m, f = m \otimes 1_-$, flux of gauged $\mathbb{Z}_2$, trivial phase.

OR

$$\mathcal{M} = \mathcal{M}_{ds} = \{1 \sim 1_+, s, \bar{s}, s\bar{s} \sim 1_-\}.$$

$s, \bar{s} = s \otimes 1_-$, flux of gauged $\mathbb{Z}_2$, nontrivial SPT

SPT is reflected in the “symmetry defects” or “flux of the gauged theory”.

Examples

Toric code with $\mathbb{Z}_2$ symmetry
(no interaction between symmetry and topological order)

$$\mathcal{E} = \mathrm{Rep}(\mathbb{Z}_2) = \{1_+, 1_-\}.$$

$$\mathcal{C} = \mathrm{Rep}(\mathbb{Z}_2) \boxtimes \mathcal{M}_{tc} = \{1_+, 1_-\} \times \{1, e, m, f\}.$$

$$\mathcal{M} = \mathcal{M}_{tc} \boxtimes \mathcal{M}_{tc} \quad \text{OR} \quad \mathcal{M} = \mathcal{M}_{ds} \boxtimes \mathcal{M}_{tc}$$

Toric code with e, m exchange $\mathbb{Z}_2$ symmetry

$$\mathcal{E} = \mathrm{Rep}(\mathbb{Z}_2) = \{1_+, 1_-\}.$$

$$\mathcal{C} = \{1_+, 1_-, f_+, f_-, \tau \sim e \oplus m\}.$$

$$\mathcal{M} \sim \text{Ising} \boxtimes \overline{\text{Ising}}, \text{two versions.}$$

Two $\mathcal{M}$ differ by the stacking of non-trivial $\mathbb{Z}_2$ SPT phase. A general theorem

Table of Topological Phases

List $\mathcal{C}$ in terms of anyon spectrum:

$\mathbb{Z}_2^f$ symmetry "fermion phases with no symmetry"

N_c^F	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$
2_0^F	$1, 1$	$0, \frac{1}{2}$
4_0^F	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$
$4_{1/5}^F$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$
$4_{-1/5}^F$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$
$4_{1/4}^F$	$1, 1, \zeta_6^2, \zeta_6^2$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$
6_0^F	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$
6_0^F	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{16}, \frac{7}{16}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$
6_0^F	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{3}{16}, \frac{5}{16}$
$6_{1/7}^F$	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{5}{14}, -\frac{1}{7}, -\frac{3}{14}, \frac{2}{7}$
$6_{-1/7}^F$	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{5}{14}, \frac{1}{7}, \frac{3}{14}, -\frac{2}{7}$
6_0^F	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{1}{2}$
6_0^F	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2}$

$$\zeta_n^m = \frac{\sin[\pi(m+1)/(n+2)]}{\sin[\pi/(n+2)]}$$

N — number of anyon types; "rank"
d_i — quantum dimension "internal degrees of freedom"
s_i — topological spin "internal angular momentum mod 1"

Theorem

Bulk excitations $\mathcal{C}$ determine topological phases up to invertible ones.

Stacking Operation

Consider stacking two layers of topological phases $\mathcal{A}$ and $\mathcal{B}$ with the same symmetry G to construct a new one $\mathcal{A} \boxtimes \mathcal{B}$.

- Before adding interactions between layers, the two layer system $\mathcal{A} \boxtimes \mathcal{B}$ has a larger symmetry $G \times G$.
- Allow inter-layer local interactions that preserves only the subgroup G of $G \times G$ via embedding $g \mapsto (g, g)$. This way the two-layer system remain with the same symmetry G. Denote such stacking by $\mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{B}$.
- The stacking $\boxtimes_{\mathcal{E}}$ is obviously associative and commutative.
- There is always a unit $\mathcal{I}_{\mathcal{E}}$ symmetric product state

$$\mathcal{I}_{\mathcal{E}} \boxtimes_{\mathcal{E}} \mathcal{A} = \mathcal{A} = \mathcal{A} \boxtimes_{\mathcal{E}} \mathcal{I}_{\mathcal{E}}.$$

- All topological phases with symmetry $\mathcal{E}$ form a commutative monad under stacking.

Invertible Phases

- $\mathcal{A}$ is invertible if there exists $\overline{\mathcal{A}}$ such that

$$\mathcal{A} \boxtimes_{\mathcal{E}} \overline{\mathcal{A}} = \mathcal{I}_{\mathcal{E}}.$$

- All invertible phases form an abelian group $\mathbf{Inv}_{\mathcal{E}}$.
- Consider the chiral central charge of the edge states. Taking central charge gives a group homomorphism from invertible phases to rational numbers:

$$c : \mathbf{Inv}_{\mathcal{E}} \to \mathbb{Q}.$$

- For a given symmetry $\mathcal{E}$, there is a smallest positive central charge $c^{\mathcal{E}}_{\min}$, which is equivalent to $c(\mathbf{Inv}_{\mathcal{E}}) = c^{\mathcal{E}}_{\min}\mathbb{Z}$.

Invertible Phases

- The kernel non-chiral symmetric invertible phases are the SPT phases $\mathbf{SPT}_{\mathcal{E}} = \ker c$. Thus we have the central extension

$$0 \to \mathbf{SPT}_{\mathcal{E}} \to \mathbf{Inv}_{\mathcal{E}} \to c^{\mathcal{E}}_{\min}\mathbb{Z} \to 0.$$

  Since $H^2(\mathbb{Z}, M) = 0$ for any abelian group M, the above must be a trivial extension

$$\mathbf{Inv}_{\mathcal{E}} = \mathbf{SPT}_{\mathcal{E}} \times c^{\mathcal{E}}_{\min}\mathbb{Z}.$$

- Will give the formula to compute $\mathbf{SPT}_{\mathcal{E}}$ and $c^{\mathcal{E}}_{\min}$.

Local Excitations under Stacking

- The symmetry is preserved by the stacking $\boxtimes_{\mathcal{E}}$, which means we should have $\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{E} = \mathcal{E}$.
- The embedding $g \mapsto (g, g)$ automatically induces a braided monoidal functor

$$\mathrm{Rep}(G) \boxtimes \mathrm{Rep}(G) = \mathrm{Rep}(G \times G) \to \mathrm{Rep}(G),$$
$$x \boxtimes y \mapsto x \otimes y,$$

  which is just taking tensor product of representions from two layers.
- We want a purely categorical description no G involved that is extendable from $\mathcal{E}$ to $\mathcal{C}$ and $\mathcal{M}$.

Local Excitations under Stacking

- Consider the tensor functor

$$\otimes : \mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E},$$
$$x \boxtimes y \mapsto x \otimes y.$$

Let its right adjoint be R, $L_{\mathcal{E}} := R(\mathbf{1}) \cong \oplus_i i \boxtimes i^*$ has a canonical structure of condensable algebra.

$L_{\mathrm{Rep}(G)} = \mathrm{Fun}[(G \times G)/G]$.

- $\mathcal{E}$ is obtained from $\mathcal{E} \boxtimes \mathcal{E}$ by condensing this $L_{\mathcal{E}}$.
- Mathematically, taking the local modules representations of $L_{\mathcal{E}}$ in $\mathcal{E} \boxtimes \mathcal{E}$

$$(\mathcal{E} \boxtimes \mathcal{E})^0_{L_{\mathcal{E}}} = \mathcal{E}.$$

So we define

$$\mathcal{E} \boxtimes_{\mathcal{E}} \mathcal{E} := (\mathcal{E} \boxtimes \mathcal{E})^0_{L_{\mathcal{E}}}.$$

Stacking of $\mathcal{C}$ and $\mathcal{M}$

- Easy to extend. For $\mathcal{E} \subset \mathcal{C}_1 \subset \mathcal{M}_1$ and $\mathcal{E} \subset \mathcal{C}_2 \subset \mathcal{M}_2$, naturally

$$L_{\mathcal{E}} \in \mathcal{E} \boxtimes \mathcal{E} \subset \mathcal{C}_1 \boxtimes \mathcal{C}_2 \subset \mathcal{M}_1 \boxtimes \mathcal{M}_2.$$

- So we just take local modules in the larger categories

condense $L_{\mathcal{E}}$ in larger categories

$$\mathcal{C}_1 \boxtimes_{\mathcal{E}} \mathcal{C}_2 := (\mathcal{C}_1 \boxtimes \mathcal{C}_2)^0_{L_{\mathcal{E}}}, \quad \mathcal{M}_1 \boxtimes_{\mathcal{E}} \mathcal{M}_2 := (\mathcal{M}_1 \boxtimes \mathcal{M}_2)^0_{L_{\mathcal{E}}}.$$

- The central charges just add up upon stacking.

Group Structures of Modular Extensions

Theorem

Under the stacking $\boxtimes_{\mathcal{E}}$, modular extensions of $\mathcal{E}$, $\mathcal{M}_{ext}(\mathcal{E})$, form an finite abelian group.

$$\mathcal{M}_{ext}(\mathcal{E}) = \mathbf{Inv}_{\mathcal{E}}/8\mathbb{Z} = \mathbf{SPT}_{\mathcal{E}} \times c^{\mathcal{E}}_{\min}\mathbb{Z}/8\mathbb{Z}.$$

A phase $\mathcal{E} \subset \mathcal{C} \subset \mathcal{M}$ is invertible if and only if every excitation is local $\mathcal{E} = \mathcal{C}$.

Theorem

The modular extensions of a $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, $\mathcal{M}_{ext}(\mathcal{C})$, if exist, form an $\mathcal{M}_{ext}(\mathcal{E})$-torsor.

$\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$ alone fixes the phase up to invertible ones.

Bosonic and Fermionic Invertible Phases

$$\mathcal{M}_{ext}(\mathcal{E}) = \mathbf{Inv}_{\mathcal{E}}/8\mathbb{Z} = \mathbf{SPT}_{\mathcal{E}} \times c_{\min}^{\mathcal{E}}\mathbb{Z}/8\mathbb{Z}.$$

Bosonic invertible phase

The abelian group $\mathcal{M}_{ext}(\mathrm{Rep}(G))$, modular extensions of $\mathrm{Rep}(G)$ is isomorphic to $H^3(G, U(1))$. Consistent with known classification of bosonic SPT.
All modular extensions of $\mathrm{Rep}(G)$ have central charge $0 \mod 8$.
$c_{\min}^{\mathrm{Rep}(G)} = 8$.

Fermionic invertible phase

Fermionic SPTs and invertible topological orders in 2+1D are given by the group $\mathcal{M}_{ext}(\mathrm{sRep}(G^f))$. The zero central charge subgroup gives the SPTs and $c_{\min}^{\mathrm{sRep}(G)} = 1/2$ or 1 can also be extracted.

Summary and Outlook

2+1D Topological Phases with Symmetry[1]

$$G \subset G_H, \quad \mathcal{E} \subset \mathcal{C} \subset \mathcal{M}, \quad c$$

	1+1D	2+1D	3+1D
Symmetry breaking	⊘✓	⊘✓[1]	✓
SPT			✓
TO with symmetry (SET)	×		✓
Topological order	×		✓[2]
Fracton topological phase	×	×	✓
???	×	×	???

3+1D Topological Order[2]

Gauging: 3+1D SPT → 3+1D topological order

3+1D topological orders are all gauged SPTs.