Quantum Current

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Review classical current

- The electric charge is a conserved quantity.
- Classically, we think that the electric charge is a continuous quantity and talk about the charge density ρ. The global charge $Q = \int \rho$ is conserved.
- If we divide the whole system into two parts A and B, and denote $Q_{\mathrm{A}} = \int_{\mathrm{A}} \rho$ and $Q_{\mathrm{B}} = \int_{\mathrm{B}} \rho$, then

  $$0 = \Delta Q = \Delta Q_{\mathrm{A}} + \Delta Q_{\mathrm{B}} \implies \Delta Q_{\mathrm{A}} = -\Delta Q_{\mathrm{B}}.$$

  The change of charge in one subsystem must compensate that in the other. When the charge in A increases, there must be charge flowing from B to A, i.e., a **current**.
- One can further consider the current density $\boldsymbol{j}$, leading to the differential equation for local conservation of charge

  $$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0.$$

Issues with classical current

- The existing notion of current in quantum mechanics does not seriously deal with symmetry charge

$$\boldsymbol{j} = e \times \text{probability current.}$$

- Only a description at statistical level.
- Requires differentiation, while electric charge is discrete instead of continuous.
- Requires *usual addition* of symmetry charge. Fails for, e.g., angular momentum (as charge of $SU(2)$), $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.
- Expect to have a description of **exact** local conservation of charge at the level of **quantum states and operators**, which applies to **any symmetry group**.
- By-product: natural physical interpretation to the **categorical symmetry**.

Setup: symmetry charge and symmetric operators

Let the symmetry group be G. The representation category $\operatorname{Rep} G =: \mathcal{C}$ encodes the data of symmetry charges and symmetric operators:

- Objects $(V, \rho : G \to GL(V))$ are group representations, physically symmetry charges;
- A morphism $f : (V, \rho) \to (W, \tau)$ is a linear map $f : V \to W$ that commutes with group actions $f\rho_g = \tau_g f, \forall g \in G$. Morphisms in $\operatorname{Rep} G$ are also called intertwiners, and physically symmetric operators.
- We will heavily use the technique of representing symmetric operators as invariant under symmetry actions tensors, or equivalently, the graphical calculus of the tensor category $\mathcal{C}$.

Setup: lattice system with onsite symmetry

Fix a set L of lattice sites and a group G. A quantum system with onsite symmetry G, on L, consists of

- For each subset $\mathrm{K} \subset \mathrm{L}$, there is a Hilbert space $\mathcal{H}_{\mathrm{K}}$ which carries a group representation $(\mathcal{H}_{\mathrm{K}}, \rho^{\mathrm{K}})$;
- A Hermitian operator (the total Hamiltonian) H on $\mathcal{H}_{\mathrm{L}}$,

such that

- For any two disjoint subsets K_1 and K_2, the representation associated to their disjoint union is the tensor product of those associated to K_1 and K_2

$$(\mathcal{H}_{\mathrm{K}_1 \coprod \mathrm{K}_2}, \rho^{\mathrm{K}_1 \coprod \mathrm{K}_2}) = (\mathcal{H}_{\mathrm{K}_1}, \rho^{\mathrm{K}_1}) \otimes (\mathcal{H}_{\mathrm{K}_2}, \rho^{\mathrm{K}_2});$$

- The total Hamiltonian has the form

$$H = \sum_{\mathrm{K} \subset \mathrm{L}} H_{\mathrm{K}},$$

  where H_{K} is a symmetric operator supported on K

Every symmetric operator carries a symmetry charge

Let's consider a bipartite system,

$$\mathcal{H} = \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}},$$

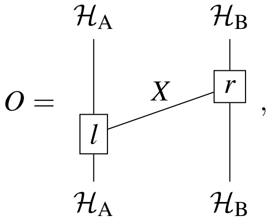
with symmetry actions $\rho_g^i \in GL(\mathcal{H}_i), i = \mathrm{A}, \mathrm{B}$ and $\rho_g = \rho_g^{\mathrm{A}} \otimes \rho_g^{\mathrm{B}} =: \rho_g^{\mathrm{A}} \rho_g^{\mathrm{B}}$.
A symmetric operator acting on the total space is by definition an operator $O : \mathcal{H} \to \mathcal{H}$ that commutes with symmetry actions

$$\rho_g O \rho_g^{-1} = O, \quad \forall g \in G.$$

In general, O does not commute with "partial" group actions ρ^{A} or ρ^{B}, and the charge within A or B is not conserved.
Indeed, O can transport symmetry charge between A and B.

Every symmetric operator carries a symmetry charge

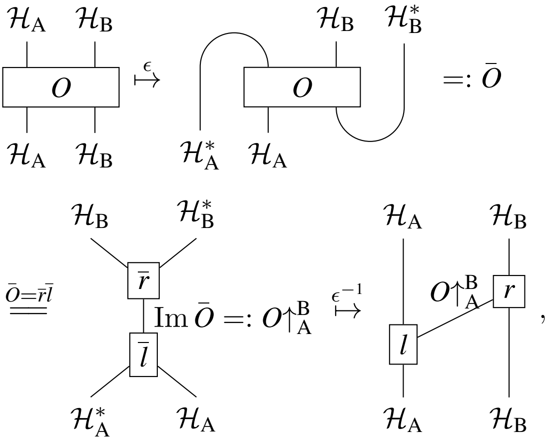
- We are tempted to represent O by the following tensor



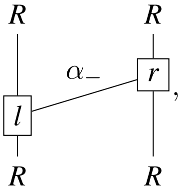
  and interpret X as the symmetry charge transported by O from subsystem A to subsystem B.
- l, r describe how the charge X leaves A and arrives at B.
- However, a large enough representation X large bond dimension can always do the job to represent O . We need to find the smallest X.

Extract the transported charge

The algorithm to extract the symmetry charge transported by O, denoted by $O\uparrow_{\mathrm{A}}^{\mathrm{B}}$, is as follows essentially a symmetry-preserving SVD (SSVD):



Let $G = \mathbb{Z}_2 := \{1, \zeta\}$. Consider the regular representation R on a qubit with nontrivial action $\rho_\zeta = \sigma_x$. The symmetric operator $O = \sigma_z^{\mathrm{A}} \sigma_z^{\mathrm{B}}$ transports an odd $\mathbb{Z}_2$ charge:

$\sigma_z^{\mathrm{A}}\ \sigma_z^{\mathrm{B}} =$ 

,

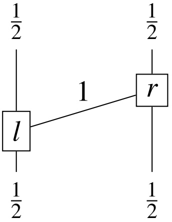
$$l_{|1\rangle}^{|1\rangle|-\rangle} = -l_{|\zeta\rangle}^{|\zeta\rangle|-\rangle} = 1 = r_{|-\rangle|1\rangle}^{|1\rangle} = -r_{|-\rangle|\zeta\rangle}^{|\zeta\rangle},$$ others are 0

This result is physically easy to understand: the operator σ_z flips the $\mathbb{Z}_2$ charge, and $O = \sigma_z^{\mathrm{A}} \sigma_z^{\mathrm{B}}$ flips the $\mathbb{Z}_2$ charges of both sites together, which is the same as moving an odd $\mathbb{Z}_2$ charge from one site to the other.

Example $SU(2)$

Let $G = SU(2)$. The Heisenberg interaction transports an angular momentum of spin 1:

$$\frac{1}{3}\vec{\sigma}^{\mathrm{A}} \cdot \vec{\sigma}^{\mathrm{B}} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

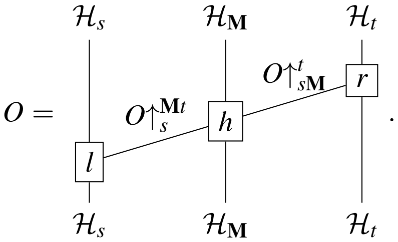


.

where l, r are intertwiners given by the Clebsch-Gordan coefficients.

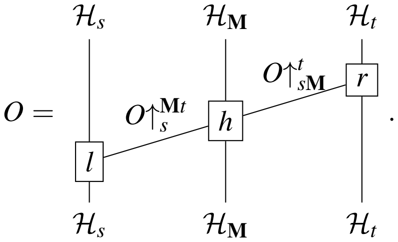
Flow of symmetry charge

Now consider a tripartite subsystem $\mathcal{H}_s \otimes \mathcal{H}_{\mathbf{M}} \otimes \mathcal{H}_t$.
For symmetric operator O acting on this subsystem, we can similarly extract the symmetry charge transported from s to $\mathbf{M}$ to t, by repeatedly performing the SSVD process:



With such graphical representation, we can read out two transported charges $O{\uparrow}^{\mathbf{M}t}_{s}$ and $O{\uparrow}^{t}_{s\mathbf{M}}$, together with an invariant tensor h telling us how the charge flows through $\mathbf{M}$.

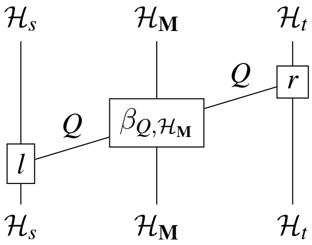
Flow of symmetry charge



Supposing that we want the operator O to only transport charge from s to t with no other effect on the intermediate region $\mathbf{M}$, and moreover, the intermediate region $\mathbf{M}$ can be arbitrary, we arrive at the following definition of quantum current.

Quantum Current

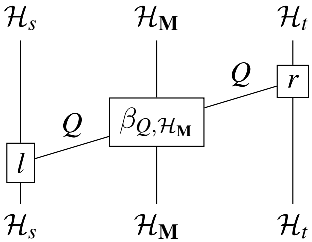
In a lattice system with onsite symmetry, $(\mathrm{L}, \mathcal{H}_\mathrm{K})$, (the Hamiltonian can be arbitrary), a quantum current (Q, β) is a collection symmetric operators of the following form



where

- $s, t \in \mathrm{L}$ are called the source and target sites;
- $\mathbf{M} \subset \mathrm{L}$ is an arbitrary intermediate subregion, $\mathcal{H}_\mathbf{M} = \bigotimes_{i \in \mathbf{M}} \mathcal{H}_i$;

Quantum Current (continued)

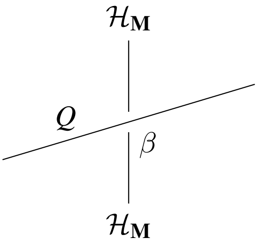


- Q is the symmetry charge transported from s to t;
- l, r are arbitrary intertwiners in $\mathrm{Hom}(\mathcal{H}_s, \mathcal{H}_s \otimes Q)$ and $\mathrm{Hom}(Q \otimes \mathcal{H}_t, \mathcal{H}_t)$ respectively, and called source and target intertwiners;
- β is a collection of invertible intertwiners $\{\beta_{Q,V} : Q \otimes V \to V \otimes Q\}$ for any $V \in \mathcal{C}$;

Quantum Current (continued)

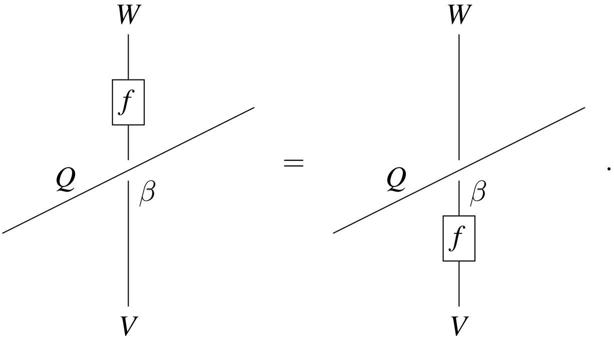
Let's focus on the intertwiners β:

$\beta_{Q,\mathcal{H}_{\mathbf{M}}} =$



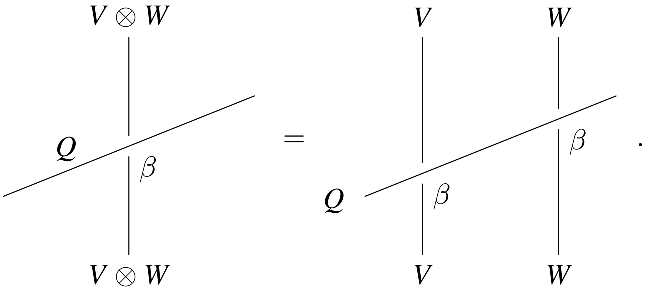
where for simplicity we omitted the subscript of β in the graph which can be unambiguously read out from the decorations on the legs. We also draw the β node intuitively as a crossing-over. β must be compatible with the arbitrary choice of $\mathbf{M}$, which turns out to be the following conditions:

(1) β commutes with local symmetric operators (naturality): for any $f : V \to W$ in $\mathcal{C}$

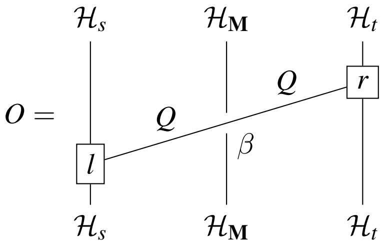


Quantum Current (continued)

(2) β is compatible with any bipartition of the intermediate region:
for any $V, W \in \mathcal{C}$,



These two conditions for β are exactly the axioms of **half-braiding**. A quantum current corresponds to an object in the Drinfeld center $Z_1(\mathcal{C})$.



Theorem

Condensed quantum currents form Lagrangian algebras.

- A symmetric operator $O \in (Q, \beta)$ with a fixed choice of $s, \mathbf{M}, t, l, r$ is called a realization of the quantum current (Q, β).
- Suppose a Hamiltonian $H = \sum_\mathrm{K} H_\mathrm{K}$ is given. A non-zero realization $O \in (Q, \beta)$, with non-empty $\mathbf{M}$, is called condensed if $OH = HO$. A quantum current (Q, β) is called condensed if it has a realization that is condensed.

Renormalization in 1+1D lattice system

- To test the above ideas, we developed a rigorous scheme for renormalization in 1+1D lattice system with onsite symmetry G.
- We find that gapped fixed-points correspond to isometric Frobenius algebras (A, m, η) in $\mathcal{C} = \mathrm{Rep}\, G$.
- We figured out the fixed-point Hamiltonians and ground states on an infinite chain, on an half-infinite chain with boundary, and on a chain with defects.
- These data are organized by the Morita theory in $\mathcal{C}$, i.e., the 2-category of algebras, bimodules and bimodule maps in $\mathcal{C}$.

Isometric Frobenius algebra

An isometric Frobenius algebra in $\mathcal{C} = \mathrm{Rep}\, G$ is a group representation A together with two intertwiners $m : A \otimes A \to A$ $a \cdot b := m(a \otimes b)$, $g(a \cdot b) = (ga) \cdot (gb)$, and $\eta : \mathbb{C} \to A$, satisfying associativity

$$m(\mathrm{id}_A \otimes m) = m(m \otimes \mathrm{id}_A), \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

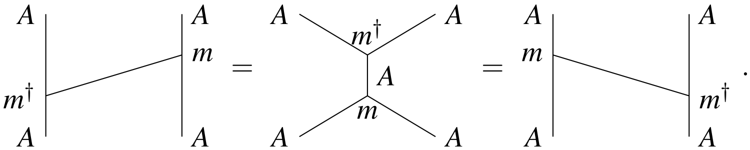
unitality

$$m(\mathrm{id}_A \otimes \eta) = \mathrm{id}_A = m(\eta \otimes \mathrm{id}_A), \quad a \cdot \eta(1) = a = \eta(1) \cdot a$$

and isometric condition

$$mm^\dagger = 1.$$

The Frobenius condition automatically follows:



1+1D fixed-point model from Frobenius algebra

Using the Frobenius algebra (A, m, η) we define a 1+1D fixed-point lattice model:

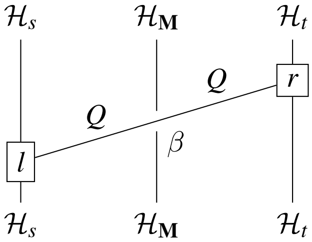
- The local Hilbert space on each site is A;
- The Hamiltonian involves only nearest-neighbor interaction: $m^\dagger m = A \otimes A \xrightarrow{m} A \xrightarrow{m^\dagger} A \otimes A$;

$$H = -\sum_i (m^\dagger m)_i.$$

- $m^\dagger m$ are commuting projectors, and thus the model is exactly solvable and gapped.
- The excitations in this model are described by the A-A-bimodules in $\mathcal{C}$, denoted by ${}_A\mathcal{C}_A$.

Condensed Quantum Currents

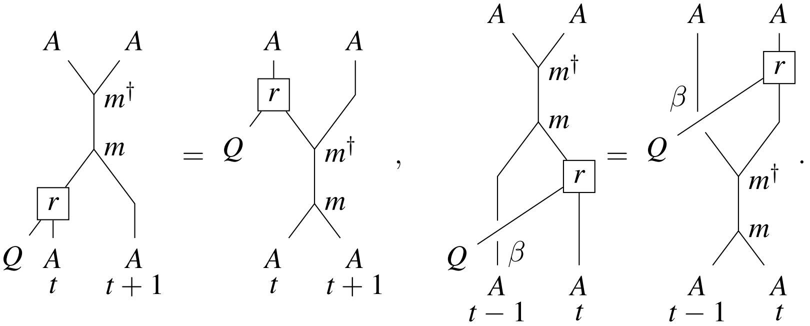
Now suppose the realization O of quantum current (Q, β)

$O =$ 

is condensed in the model $H = -\sum_i (m^\dagger m)_i$, i.e., $OH = HO$. Since O automatically commutes with $m^\dagger m$ terms supported within $\mathbf{M}$, we only need to check around the s, t sites.

Condensed Quantum Currents (continued)

Conditions for O to commute with H around s and around t turn out to be equivalent. We depict the condition around t.
(Q, β) is condensed if there is non-zero $r \in \mathrm{Hom}(Q \otimes A, A)$ such that



The two conditions are equivalent to that $r \in \mathrm{Hom}(Q \otimes A, A)$ is an A-A-bimodule map.

Condensed Quantum Currents (continued)

Physically, the non-zero A-A-bimodules maps from $Q \otimes A$ to A counts the ways how quantum current (Q, β) is condensed. Using the internal hom adjunction

$$\mathrm{Hom}_{{}_A\mathcal{C}_A}(Q \otimes A, A) \cong \mathrm{Hom}_{Z_1(\mathcal{C})}((Q, \beta), [A, A]),$$

We conclude

Theorem

Given a Frobenius algebra (A, m, η) in $\mathcal{C}$. The universal quantum current condensed in $H = -\sum_i (m^\dagger m)_i$ is $[A, A]$, which is a Lagrangian algebra in $Z_1(\mathcal{C})$ DGNO, arXiv:1009.2117 . The excitations are related to the condensed quantum currents via

$${}_A\mathcal{C}_A \cong Z_1(\mathcal{C})_{[A,A]}.$$

Thanks for attention!