The Modular Extension Characterization of SPT/SET phases

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T. Lan, L. Kong, and X.-G. Wen, Phys. Rev. B 94, 155113 (2016), arXiv:1507.04673.
T. Lan, L. Kong, and X.-G. Wen, Phys. Rev. B 95, 235140 (2017), arXiv:1602.05946.
T. Lan, L. Kong, and X.-G. Wen, Commun. Math. Phys. 351, 709 (2016), arXiv:1602.05936.

Overview

A "Functor" **PHYSICS** $\to$ **MATHEMATICS**

Topological phase	$\mapsto$	Braided fusion category
Particle (excitation)	$\mapsto$	Object
Process (fusion, braiding, ...)	$\mapsto$	Morphism
Stacking		? ⊠ when no symmetry

I will explain

- the structure of symmetry on both sides,
- the structure of stacking on both sides.

Surprisingly, these are enough for us to establish a description of topological phases with symmetry (i.e. SPT/SET phases).

Quantum Phase

To define a quantum phase, we need at least

- A quantum system (H, V) with Hamiltonian H and Hilbert space V;
- A proper notion of locality and thermodynamic limit: Local Hilbert space, local operator, $V = \otimes_l V_l$, $H = \sum_l H_l$, the total number of sites goes to ∞ ...
- A proper characterization of phase transition under thermodynamic limit gap closing, for example;
- Equivalence relation of quantum systems $(H, V) \xleftrightarrow{\text{no phase transition}} (H', V')$;
- A quantum phase is then the equivalence class $[(H, V)]$.

Fortunately, to proceed, not every detail on the **PHYSICS** side is needed.

Topological phase $\mapsto$ braided fusion category, ...

Symmetry

The traditional characterization of a quantum system (H, V) with symmetry

- Symmetry group G,
- Symmetry (group) action $\rho : G \to GL(V)$, V is a representation of G
- H is symmetric (an intertwiner): $H\rho_g = \rho_g H$.

However, this perspective is not convenient. Instead, note that

- V decomposes to direct sum of irreducible representations symmetry charges;
- Superselection rule: no evolution between different irreducible representations because H is an intertwiner and of Schur's Lemma, evolutions are also intertwiners and symmetry charge is conserved;

Symmetric Phase

- Equivalence relation between quantum systems with symmetry $(H, V, \rho) \xleftrightarrow[\text{respect symmetry}]{\text{no phase transition}} (H', V', \rho')$;
- Symmetric phase is the equivalence class $[(H, V, \rho)]$.
- It is reasonable to expect that all representations of G could appear in a symmetric phase $[(H, V, \rho)]$.
- The category of representation and intertwiners, $\mathrm{Rep}(G)$, **is** a symmetric braided fusion category, the invariant we seek for.
- $\mathrm{Rep}(G)$ and G determines each other by Tannaka duality.

Stacking

- At the quantum system level, (H, V, ρ) and (H', V', ρ') with the same symmetry G can be put side by side to form a composite system $(H \otimes H', V \otimes V')$ of the same symmetry.
- The symmetry action on the composite system is

$$\begin{aligned} G \xrightarrow{\Delta} G \times G \xrightarrow{\rho\otimes\rho'} GL(V) \otimes GL(V') = GL(V \otimes V') \\ g \mapsto \Delta(g) = (g, g) \mapsto \rho_g \otimes \rho'_g \end{aligned}$$

- The stacking of symmetric quantum phases is then obtained by taking the equivalence class (i.e. allow symmetric deformations)

$$[(H, V, \rho)] \boxtimes_G [(H', V', \rho')] := [(H \otimes H', V \otimes V', (\rho \otimes \rho') \circ \Delta)].$$

Stacking

The stacking operation has some obvious structures

- It is associative and commutative

$$([H_1] \boxtimes_G [H_2]) \boxtimes_G [H_3] = [H_1] \boxtimes_G ([H_2] \boxtimes_G [H_3]),$$

$$[H_1] \boxtimes_G [H_2] = [H_2] \boxtimes_G [H_1].$$

- Identity exists, i.e., the trivial phase (product state) $I := [H_0 = -\sum_l P_l]$, where P_l are local projections.

$$I \boxtimes_G [H] = [H] \boxtimes_G I = [H].$$

- The phases thus form a commutative monoid under stacking.
- For 2+1D phases, the chiral central charge of the edge states adds up under stacking. Thus taking the central charge

$$[H] \mapsto c_- := c_L - c_R,$$

is a homomorphism.

Invertible and SPT/SET Phases

- We may further take the phases invertible under stacking, and they form an abelian group $\mathbf{INV}_G$.
- Among invertible phases, the symmetry protected trivial/topological (SPT) phases are those that become trivial when the symmetry is totally broken. They form an abelian group $\mathbf{SPT}_G \subset \mathbf{INV}_G$.
- Phases with both symmetry and topological order are called symmetry enriched topological (SET) phases.
- SPTs are SETs with trivial topological order.

2+1D SPT Phases

- In 2+1D it is believed that we have the following exact sequence

$$0 \to \mathbf{SPT}_G \hookrightarrow \mathbf{INV}_G \xrightarrow{c_-} c_G^{\min}\mathbb{Z} \to 0.$$

$c_G^{\min}$ is the smallest positive chiral central charge. The sequence splits since $c_G^{\min}\mathbb{Z}$ is free, $\mathbf{INV}_G = \mathbf{SPT}_G \times c_G^{\min}\mathbb{Z}$.

- For a bosonic finite unitary onsite symmetry G, we have

$$\mathbf{SPT}_G = H^3(G, U(1)).$$

X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 84, 235141 (2011), arXiv:1106.4752.
X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013), arXiv:1106.4772.
X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Science 338, 1604 (2012), arXiv:1301.0861.

This fact is simple but nontrivial enough, by which we can derive the correct characterization of SPT/SET phases.

- Also $c_G^{\min} = 8$ for the E_8 state with trivial symmetry action.

Symmetry and Stacking in terms of BFC

- We have explained the structures of symmetry and stacking on the **PHYSICS** side.
- On the **MATHEMATICS** side, we try to find the same structures, in terms of the language of braided fusion category (BFC).
- Focusing on a bosonic finite unitary onsite symmetry G, the correct mathematical structure of stacking has to recover the group $H^3(G, U(1))$.
- To implement symmetry in a braided fusion category, we need notions such as local excitation and braiding nondegeneracy.

Measurability

Principle

Things in physics must be measurable at least indirectly.

For SETs we have two means of measurement

- Remotely, by braiding;
- Locally, by symmetry.

Local excitation

A local excitation is one that can be created or annihilated by local operators.

Local excitations cannot be measured remotely; they have to be measured by symmetry:

category of local excitations $=$ category of symmetry charges.

We are interested in the case that the category of symmetry charges is $\mathrm{Rep}(G)$ in this talk.

BFC with Symmetry

Braiding nondegeneracy

Things that cannot be measured by braiding have to be local.

In a BFC $\mathcal{C}$, the Müger center $Z_2(\mathcal{C})$, namely the full subcategory of objects x satisfying denote by c_{xy} the braiding in $\mathcal{C}$

$$c_{yx} \circ c_{xy} = \mathrm{id}_{x \otimes y}, \ \forall y \in \mathcal{C},$$

is precisely "things that cannot be measured by braiding." Therefore, if a SET with symmetry G is assigned to a BFC $\mathcal{C}$, we must have

$$Z_2(\mathcal{C}) = \mathrm{Rep}(G).$$

More precisely, the embedding (fully faithful braided monoidal functor) should be specified,

$$\mathrm{Rep}(G) \hookrightarrow \mathcal{C},$$

whose image is equivalent to $Z_2(\mathcal{C})$.

Stacking of BFC with Symmetry

The symmetry of a two-layer system is given by the diagonal map

$$\Delta : G \to G \times G, \ \Delta(g) = (g, g),$$

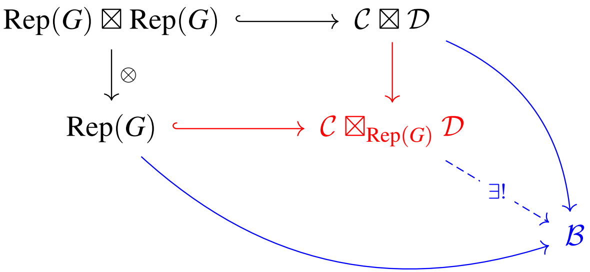
which induces the braided functor on representations

$$\mathrm{Rep}(G) \xleftarrow{\otimes} \mathrm{Rep}(G) \boxtimes \mathrm{Rep}(G) = \mathrm{Rep}(G \times G).$$

The above can be interpreted as breaking the larger two-layer symmetry $G \times G$ down to G, or adding up the symmetry charges in two layers.

Stacking of BFC with Symmetry

- Now suppose that $\mathcal{C}$ and $\mathcal{D}$ describe two SETs, namely, they both contain $\mathrm{Rep}(G)$ as their Müger center.
- $\mathcal{C} \boxtimes \mathcal{D}$ is then a BFC that contains $\mathrm{Rep}(G) \boxtimes \mathrm{Rep}(G)$ as its Müger center, a SET with the larger symmetry $G \times G$.
- We should break $G \times G$ to G in $\mathcal{C} \boxtimes \mathcal{D}$ to obtain the correct stacking, which may be defined via universal property, by the pushout in the 2-category of BFC.



Stacking of BFC with Symmetry

$\mathcal{C} \boxtimes_{\mathrm{Rep}(G)} \mathcal{D}$ can be constructed explicitly using the techniques of algebra objects in BFC, or physically, anyon condensation.

- In $\mathrm{Rep}(G) \boxtimes \mathrm{Rep}(G)$ there is canonical algebra object $A := \mathrm{Fun}(G \times G/\Delta(G))$, the functions on the cosets. The category of local modules over A in $\mathrm{Rep}(G) \boxtimes \mathrm{Rep}(G)$ is canonically equivalent to $\mathrm{Rep}(G)$.
- Since $\mathrm{Rep}(G)$ embeds into $\mathcal{C}$ and $\mathcal{D}$, the similar thing can be done in $\mathcal{C} \boxtimes \mathcal{D}$. Just take the category of local modules over A in the larger category $\mathcal{C} \boxtimes \mathcal{D}$ instead of $\mathrm{Rep}(G) \boxtimes \mathrm{Rep}(G)$, and we obtain the desired $\mathcal{C} \boxtimes_{\mathrm{Rep}(G)} \mathcal{D}$, which automatically contains $\mathrm{Rep}(G)$ and satisfies the universal property.
- Physically, by condensing A we break the symmetry from $G \times G$ to G.

Missing Structures

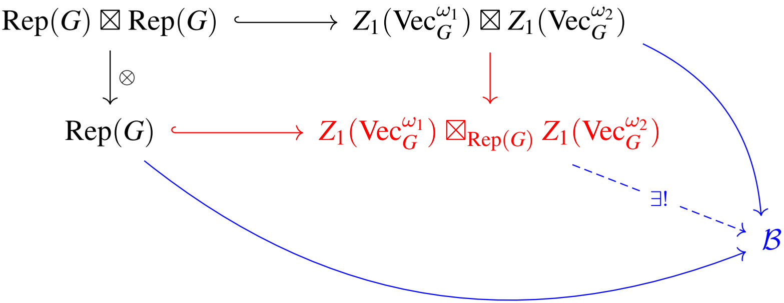
- Under such definition of stacking, $\mathrm{Rep}(G)$ is the identity under stacking, and the invertible things turn out to be only $\mathrm{Rep}(G)$.
- We miss the information of SPT and invertible phases.
- We need to equip $\mathrm{Rep}(G)$ with some additional structures to capture SPT phases.
- The answer turns out to be minimal modular extensions. Instead of postulate it directly, let's search for the $H^3(G, U(1))$ data in BFCs.

H^3 in BFCs

- An obvious example is the G-graded vector spaces, as fusion categories.
- When the fusion rules form the group G, the associativity constraints are classified by 3-cocycles $\omega \in H^3(G, U(1))$.
- Denote such a fusion category by Vec_G^ω: the isomorphism classes of simple objects are labeled by $g \in G$, tensor product is given by $g \otimes h \cong gh$, and the associator is $(g \otimes h) \otimes l \xrightarrow{\omega(g,h,l)} g \otimes (h \otimes l)$.
- To obtain a BFC, we can just take the Drinfeld center $Z_1(\mathrm{Vec}_G^\omega)$.

H^3 in BFCs

It is a straightforward exercise to check that $\mathrm{Rep}(G)$ is contained in $Z_1(\mathrm{Vec}_G^\omega)$. Moreover, if we define the "stacking" of $Z_1(\mathrm{Vec}_G^\omega)$ similarly as the pushout



One can check $Z_1(\mathrm{Vec}_G^{\omega_1}) \boxtimes_{\mathrm{Rep}(G)} Z_1(\mathrm{Vec}_G^{\omega_2}) = Z_1(\mathrm{Vec}_G^{\omega_1+\omega_2})$.

H^3 in BFCs

It is thus a reasonable guess that $Z_1(\mathrm{Vec}_G^\omega)$ is the structure needed for characterizing SPT phases. Moreover, they should be the "only solutions." $Z_1(\mathrm{Vec}_G^\omega)$ has the following properties

- It is a modular tensor category (MTC, nondegenerate BFC, Müger center is trivial);
- It is the smallest MTC that contains $\mathrm{Rep}(G)$, in the sense of $\dim(Z_1(\mathrm{Vec}_G^\omega)) = \dim(\mathrm{Rep}(G))^2 = |G|^2$.

It can be proved that a MTC $\mathcal{M}$ containing $\mathrm{Rep}(G)$ whose total quantum dimension is $|G|^2$ must be equivalent to the Drinfeld center of a fusion category with fusion rule G, that is $\mathcal{M} = Z_1(\mathrm{Vec}_G^\omega)$ for some $\omega \in H^3(G, U(1))$ by condensing the Lagrangian algebra $\mathrm{Fun}(G)$ in $\mathcal{M}$.

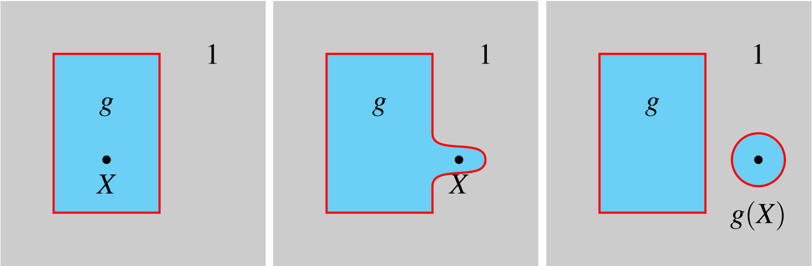
The physical idea of Gauging

Physically, to probe the SPT phases, one can also gauge the symmetry.

M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012), arXiv:1202.3120.

To measure the symmetry charge of local excitation, one can examine

symmetry action = "braiding" with symmetry defect.



The physical idea of Gauging

- Thus, by promoting the global symmetry to local symmetry, and the symmetry defects to excitations, i.e., gauging couple the original system to a gauge field, we can measure the symmetry charges via braiding in the gauged theory.
- From the categorical point of view, gauging means adding excitations which braid nontrivially with local excitations, until the whole theory is braiding nondegenerate everything is detectable.
- The definition of minimal modular extension follows directly from such idea.

Minimal Modular Extension

We are ready to state the most general formulation, which has essentially the same ingredients as the H^3 case:

- A symmetry is characterized by the category of symmetry charge, a symmetric fusion category $\mathcal{E}$ includes fermionic symmetry;
- The category of excitations of a SET with symmetry $\mathcal{E}$ is characterized by a BFC $\mathcal{C}$ with $\mathcal{E} \hookrightarrow \mathcal{C}$ such that $\mathcal{E}$ is equivalent to the Müger center of $\mathcal{C}$;
- A minimal modular extension of $\mathcal{E} \hookrightarrow \mathcal{C}$ is a MTC $\mathcal{M}$ with $\mathcal{C} \hookrightarrow \mathcal{M}$ such that the Müger centralizer of $\mathcal{E}$ in $\mathcal{M}$ the excitations in $\mathcal{M}$ that can not measure $\mathcal{E}$ via braiding, which by the above point is at least $\mathcal{C}$ coincides with $\mathcal{C}$ in other words, all extra excitations in $\mathcal{M}$ but not in $\mathcal{C}$ braids nontrivially with $\mathcal{E}$; $\mathcal{M}$ is minimal, $\dim \mathcal{M} = \dim \mathcal{E} \dim \mathcal{C}$. $\mathcal{M}$ may be interpreted as the gauged theory, which captures more information including the SPT phases.

Stacking

The stacking of $\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}$ with $\mathcal{E} \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{N}$ is defined as the pushouts

$$\begin{array}{ccccc} \mathcal{E} \boxtimes \mathcal{E} & \hookrightarrow & \mathcal{C} \boxtimes \mathcal{D} & \hookrightarrow & \mathcal{M} \boxtimes \mathcal{N} \\ \downarrow \otimes & & \downarrow & & \downarrow \\ \mathcal{E} & \hookrightarrow & \mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D} & \hookrightarrow & \mathcal{M} \boxtimes_{\mathcal{E}} \mathcal{N} \end{array}$$

Denote by $\mathcal{M}ext(\mathcal{C})$ the set of isomorphism classes of minimal modular extensions $\mathcal{C}$:

Theorem

$\mathcal{M}ext(\mathcal{E})$ form an abelian group under stacking.
$\mathcal{M}ext(\mathcal{C})$, if non-empty, form a torsor over $\mathcal{M}ext(\mathcal{E})$.

Discussions

- Straightforward generalization to higher dimensions and higher symmetries, by considering in the higher category of braided fusion n-categories.
- An invertible phase with no symmetry has no topological excitation thus a trivial categorical description. Adding a trivial symmetry action to it gives a symmetric invertible phase. Such phases are missing in the categorical description. In 2+1D, however, all such phases are layers of E_8 with $c_- = 8k$, and MTC determines c_- modulo 8; thus it is believed by including c_- we have a full description.
- A minimal modular extension may not exist, which indicates the existence of anomaly. To make such anomaly explicit, one has to go beyond the modular extension characterization.

Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, and Hao Zheng, "Classification of topological phases with finite internal symmetries in all dimensions", J. High Energy Phys. 2020, 93 (2020), arXiv:2003.08898.