Introduction to Categorical Approach to Topological Phases in Arbitrary Dimensions

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Objectives

- Build a general framework for topological phases in arbitrary dimensions.
- Classify SET/SPT (symmetry enriched/protected topological) phases in higher dimensions.
- Understand higher symmetries.

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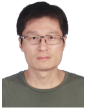

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General Framework

- Study the higher category of topological defects.
- Categorical philosophy:
  – Care about the relations between things, instead of what the thing itself is.
- Defect itself is a relation.
- Fusion of defects.
- Boundary-bulk relation.
- "Defect" in a very general sense:
  - An nD nD always mean n spacetime dimensions in this talk anomaly-free topological phase is a defect in the $(n+1)$D trivial phase.
  - No defect is understood as a trivial defect.
  - A pD topological excitation is a defect on a $(p+1)$D trivial defect.

Defect as a relation

- Suppose that A, B are nD defects, C is an $(n-1)$D defect between A, B.
- In categorical language, C is a morphism from A to B, denoted by $C : A \to B$.
- The $(n-1)$-category of all the defects between A and B is denoted by $\mathrm{Hom}(A, B)$.

Notation

- $\mathbf{1}_{n+1}$ denotes the $(n+1)$D trivial phase.
- $\mathbf{1}_n$ is the trivial defect in $\mathbf{1}_{n+1}$, also the nD trivial phase.
- $C : \mathbf{1}_{n+1} \to \mathbf{1}_{n+1}$ is an anomaly-free nD topological phase.
- $\mathcal{C} := \mathrm{End}(C) = \mathrm{Hom}(C, C)$ is all the $(n-1)$D defects in C.
- $\mathrm{id}_C : C \to C$ is the $(n-1)$D trivial defect in C.
- $\Omega\mathcal{C} := \mathrm{End}(\mathrm{id}_C) = \mathrm{Hom}(\mathrm{id}_C, \mathrm{id}_C)$, looping of $\mathcal{C}$, is the $(n-2)$D excitations in C.
- $\mathrm{id}_{\mathrm{id}_C} : \mathrm{id}_C \to \mathrm{id}_C$ is the $(n-2)$D trivial defect in C.
- $\Omega^2\mathcal{C} = \Omega\Omega\mathcal{C} := \mathrm{End}(\mathrm{id}_{\mathrm{id}_C})$ is the $(n-3)$D excitations in C.
- ...

Fusion of defects

- Two nD defects $B : A \to A$ and $C : A \to A$ can be fused to a defect $B \boxtimes_A C : A \to A$.
- Two $(n-1)$D defects $P, Q : B \to B$ on B, can fuse along B, $P \boxtimes_B Q : B \to B$.
- Two $(n-1)$D defects $P : B \to B$, $R : C \to C$ on B, C respectively, can fuse together with B, C, $P \boxtimes_A R : B \boxtimes_A C \to B \boxtimes_A C$.
- Higher codimension, more ways of fusion.

Fusion of defects

- Consistency between different ways of fusion.

Commutativity of higher codimensional excitations

For an nD phase $C : \mathbf{1}_{n+1} \to \mathbf{1}_{n+1}$:

- Codimension 1 defects $\mathcal{C} := \mathrm{End}(C) = \mathrm{Hom}(C, C)$ is a monoidal (fusion)
- Codimension 2 excitations $\Omega\mathcal{C} := \mathrm{End}(\mathrm{id}_C)$ is a braided $(n-2)$-category. Two ways of fusion
- Codimension p excitations $\Omega^{p-1}\mathcal{C}$ is a E_p-monoidal $(n-p)$-category. p ways of fusion

Boundary-bulk relation

Holographic principle

Boundary uniquely determines the bulk.

Let $C : \mathbf{1}_{n+1} \to \mathbf{1}_{n+1}$ be an nD anomaly-free phase.

- The boundary of C is a defect $B : C \to \mathbf{1}_n$.
- How C is determined from B?
- Consider the category of defects $\mathcal{C} = \mathrm{Hom}(C, C)$ and $\mathcal{B} = \mathrm{Hom}(B, B)$. $\mathcal{C}$ can be determined from $\mathcal{B}$ via two steps of constructions:

$$\mathcal{C} = \Sigma Z_1(\mathcal{B}).$$

  $Z_1(\mathcal{B})$ is the Drinfeld center of $\mathcal{B}$ and Σ denotes condensation completion.

Drinfeld center

- Bulk to boundary functor: Bring the trivial defect id_C to B will not change B. Thus codimension 2 excitations $\Omega\mathcal{C} = \mathrm{Hom}(\mathrm{id}_C, \mathrm{id}_C)$ can be brought to the boundary. There is a functor $F_B : \Omega\mathcal{C} \to \mathcal{B}$ by $F_B(X) = X \boxtimes_C \mathrm{id}_B$.
- Such functor is central: $\forall X \in \Omega\mathcal{C}$ and $Y \in \mathcal{B}$

$$F_B(X) \boxtimes_B Y \simeq Y \boxtimes_B F_B(X).$$

Drinfeld center

- Let $\mathcal{B}$ be a monoidal n-category. Its Drinfeld center (or E_1 center) $Z_1(\mathcal{B})$ is the "maximal" (limit, universal) braided monoidal n-category such that there is central functor from $Z_1(\mathcal{B})$ to $\mathcal{B}$. $Z_1(\mathcal{B})$ can be constructively defined via half-braiding, or bimodule functors $\mathrm{Hom}_{\mathcal{B}|\mathcal{B}}(\mathcal{B},\mathcal{B})$.
- In general the bulk $\Omega\mathcal{C}$ is a subcategory of the Drinfeld center $Z_1(\mathcal{B})$ of the boundary $\mathcal{B}$.
- When there is no other constraints such as symmetry, we should have $\Omega\mathcal{C} = Z_1(\mathcal{B})$.

Condensation completion

- We denote by $\Sigma\mathcal{B}$ the condensation completion of $\mathcal{B}$. Physically, the condensation completion means including all $(n+1)$D defects that can be obtained via the condensation of nD or lower defects.
- Mathematically, "completion" means including the results of all possible operations.
  - Expansion of numbers: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
  - Constructing a vector space from a given set of basis vectors. One includes all linear combinations.
- The difference is that higher category has far more operations. Vector space has one way of addition and one way of mulplication while n-category has n ways of addition (direct sum) and n ways of mulplication (fusion).

Condensation completion

- A mathematical formulation was given by Gaiotto and Johnson-Freyd. $\Sigma\mathcal{B}$ is the Karoubi (or idempotent) completion of one point delooping of $\mathcal{B}$.

D. Gaiotto, T. Johnson-Freyd, Condensations in higher categories, 2019, [arXiv:1905.09566].

- Looping $\Omega\mathcal{C} = \mathrm{Hom}(\mathrm{id}_C, \mathrm{id}_C)$.
- Delooping: Given a fusion n-category $\mathcal{B}$, the one point delooping is an $(n+1)$-category with only one object $*$ and morphisms being $\mathcal{B}$.
- Looping is the left inverse to condensation completion $\Omega\Sigma\mathcal{B} = \mathcal{B}$.
- The higher defects may not all come from the condensation of lower ones. Thus $\Sigma\Omega\mathcal{C} \subset \mathcal{C}$. $\Sigma\Omega$ is like a "projection".
- But possibly our condensation completion needs to be more general.

Boundary-bulk relation revisit

- Suppose $C : \mathbf{1}_{n+1} \to \mathbf{1}_{n+1}$ has a boundary $B : C \to \mathbf{1}_n$.
- By the holographic principle, B should uniquely determine C.
- Let $\mathcal{C} = \mathrm{Hom}(C, C)$ and $\mathcal{B} = \mathrm{Hom}(B, B)$. When there is no symmetry, $\Omega\mathcal{C}$ is determined by $\mathcal{B}$ via Drinfeld center $\Omega\mathcal{C} = Z_1(\mathcal{B})$.
- It is natural to expect that all codimension 1 defects must come from condensation of codimension 2 excitations, namely $\mathcal{C} = \Sigma\Omega\mathcal{C} = \Sigma Z_1(\mathcal{B})$.

Anomaly-free condition

For nD anomaly-free phase $C : \mathbf{1}_{n+1} \to \mathbf{1}_{n+1}$, and $\mathcal{C} = \mathrm{Hom}(C, C)$, the followings are true

1. $Z_1(\mathcal{C}) = \Omega\,\mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1}) = \mathrm{Hom}(\mathbf{1}_n, \mathbf{1}_n)$.
2. $\Omega\mathcal{C}$ is a non-degenerate braided fusion $(n-2)$-category. And $\mathcal{C} = \Sigma\Omega\mathcal{C}$, all codimension 1 defects must come from condensation of codimension 2 excitations.

Conjecture

The two statements above are equivalent to each other.

Example

(2+1)D topological order is described by non-degenerate braided fusion 1-category (namely modular tensor category if assuming unitary structure).

Condensation completion is not for free

Clearly, trivial phases are always anomaly-free.

Corollary

$$\mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1}) = \Sigma\,\mathrm{Hom}(\mathbf{1}_n, \mathbf{1}_n) = \cdots = \Sigma^n\,\mathrm{Hom}(\mathbf{1}_1, \mathbf{1}_1) = \Sigma^n\mathbb{C}.$$

- It seems our problem of understanding higher dimensional topological orders is “solved” by condensation condensation.
- But the truth is condensation completion is not for free.
- Unless we fully understand nD phases, we can in practice carry out the computation of condensation completion from $(n-1)$D to nD.

Stacking

- $\mathrm{TO}_n := \mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1})$ is the category of nD anomaly-free phases.
- Consider the category of defects in all dimensions. They form an ∞-category TO_∞. $\mathrm{TO}_n = \mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1})$ is looping "$\infty - n$" times of such ∞-category TO_∞. Therefore $\mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1})$ must be symmetric.
- Given $C, D \in \mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1})$, their fusion $C \boxtimes_{\mathbf{1}_{n+1}} D$ is nothing but the stacking of topological phases C and D.

Invertible phases

- $C : \mathbf{1}_{n+1} \to \mathbf{1}_{n+1}$ is invertible if there exists D such that $C \boxtimes_{\mathbf{1}_{n+1}} D = \mathbf{1}_n$.
- Consider the defects in C, D, under stacking we have

  $$\mathrm{End}(C) \boxtimes_{\mathbf{1}_{n+1}} \mathrm{End}(D) = \mathrm{End}(C \boxtimes_{\mathbf{1}_{n+1}} D),$$

  which implies that when C, D are invertible, we must have

  $$\mathrm{End}(C) = \mathrm{End}(D) = \mathrm{End}(\mathbf{1}_n).$$

- In other words, defects in an invertible phase are the same as those in the trivial phase.
- The defects inside a phase C, $\mathcal{C} = \mathrm{End}(C) = \mathrm{Hom}(C, C)$, is only a description up to invertible phases.
- Whey there is symmetry, there can be extra invertible phases with symmetry (e.g., SPT). These extra ones are still accesible by studing defects inside a phase.

Symmetry

- In the categorical approach, we focus on the symmetry "charges" which are excitations, instead of the symmetry operators. The equivalence is guaranteed by Tannaka duality.
- Given a symmetry G, the symmetry charges are $(0+1)$D excitations, given by $\mathrm{Rep}G$ as a symmetric 1-catgory.
- The symmetry charges can condense on a line, plane,. . . , to form higher dimensional phases/defects.
- 2D defects via condensation completion $\Sigma\mathrm{Rep}G \equiv 2\mathrm{Rep}G$.

  L. Kong, Y. Tian, S. Zhou, The center of monoidal 2-categories in 3+1D Dijkgraaf-Witten Theory, Adv. Math. 360 (2020) 106928 [arXiv:1905.04644].
- 3D defects via condensation completion $\Sigma^2\mathrm{Rep}G \equiv 3\mathrm{Rep}G$.

  Since (2+1)D phases are much more complicated than (1+1)D ones, so far we do not know how to compute $3\mathrm{Rep}G$ in practice.
- . . . .

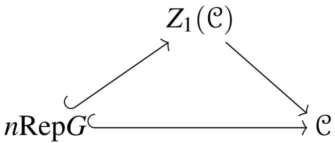
Topological phases with symmetry G

- $\mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1}) = \Sigma^{n-1}\,\mathrm{Hom}(\mathbf{1}_2, \mathbf{1}_2) = n\mathrm{Rep}G$.
- This is not the end of story though. We have to climb the ladder of dimensions and understand the phases in lower dimension, before we can compute the condensation completion.
- It is used like an assumption in proof by induction.
- Suppose we already know nD phases with symmetry G (G SET), $n\mathrm{Rep}G$. How to classify $(n+1)$D G SET? Below we give a classification up to $(n+1)$D invertible phases with no symmetry.

Topological phases with symmetry G

Given an $(n+1)$D anomaly-free phase $C : \mathbf{1}_{n+2} \to \mathbf{1}_{n+2}$, and $\mathcal{C} = \mathrm{Hom}(C, C)$ the nD defects in C. That C has symmetry G means $\mathcal{C}$ is a fusion n-category over $n\mathrm{Rep}G$, namely,

1. The symmetry charges and their higher dimensional condensation descendants are included in $\mathcal{C}$, $n\mathrm{Rep}G \hookrightarrow \mathcal{C}$.
2. The codimension 2 excitations in the bulk $\mathbf{1}_{n+2}$ of C, $\mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1}) = n\mathrm{Rep}G$, must be a subcategory of $Z_1(\mathcal{C})$, i.e., there is a braided embedding $n\mathrm{Rep}G \hookrightarrow Z_1(\mathcal{C})$.
3. $n\mathrm{Rep}G$ is local in $\mathcal{C}$. In other words, they can braid with defects in $\mathcal{C}$ in a symmetric way. Thus we require the following diagram to commute.



Topological phases with symmetry G

- In particular, take C to be the trivial phase $\mathbf{1}_{n+1}$. We know $\mathcal{C} = \mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1}) = n\mathrm{Rep}G$ is a description up to invertible phases.
- To know something about invertible phases, previous discussion suggests that we should include the canonical embedding $n\mathrm{Rep}(G) \hookrightarrow Z_1(n\mathrm{Rep}G)$ as an extra data.
- As we will later see, this allows us to see the quotient

$$\frac{\text{Invertible phases with symmetry } G}{\text{Invertible phases without symmetry}},$$

namely the SPT phases.

Topological phases with symmetry G

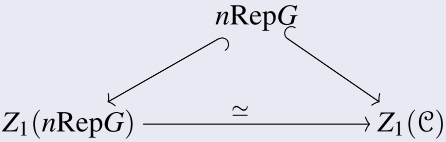
- Note that the extra data $n\mathrm{Rep}G \hookrightarrow Z_1(\mathcal{C})$ is actually in the bulk of $\mathcal{C}$.
- The bulk of trivial phase the trivial bulk is given by $n\mathrm{Rep}(G) \hookrightarrow Z_1(n\mathrm{Rep}G)$.
- For $\mathcal{C}$ to be anomaly-free, its bulk must coincide with the trivial bulk, namely we should have a braided equivalence $Z_1(n\mathrm{Rep}G) \xrightarrow{\simeq} Z_1(\mathcal{C})$ such that

$$\begin{array}{ccc} & n\mathrm{Rep}G & \\ \swarrow & & \searrow \\ Z_1(n\mathrm{Rep}G) & \xrightarrow{\simeq} & Z_1(\mathcal{C}) \end{array}$$

Topological phases with symmetry G

Theorem

The classification of G SETs in $(n+1)$D up to invertible phases with no symmetry is given by fusion n-category $\mathcal{C}$ over $n\mathrm{Rep}G$ together with a braided equivalence $Z_1(n\mathrm{Rep}G) \xrightarrow{\simeq} Z_1(\mathcal{C})$, satisfying



Liang Kong, TL, Xiao-Gang Wen, Zhi-Hao Zhang, Hao Zheng, Classification of topological phases with finite internal symmetries in all dimensions, [arXiv:2003.08898].

Topological phases with symmetry G

Corollary

In particular, when $\mathcal{C} = n\mathrm{Rep}G$, C is an invertible phase. The autoequivalences $Z_1(n\mathrm{Rep}G) \xrightarrow{\simeq} Z_1(n\mathrm{Rep}G)$ that preserve the embedding $n\mathrm{Rep}G \hookrightarrow Z_1(n\mathrm{Rep}G)$, denoted by $\mathrm{Aut}(Z_1(n\mathrm{Rep}G), n\mathrm{Rep}G)$, gives the classification of $(n+1)$D G SPTs.

Examples

- (1+1)D SPT: $\mathrm{Aut}(Z_1(\mathrm{Rep}G), \mathrm{Rep}G) = H^2(G, U(1))$.
- (2+1)D SPT: $\mathrm{Aut}(Z_1(2\mathrm{Rep}G), 2\mathrm{Rep}G) = H^3(G, U(1))$.
- The higher dimensional examples are not easy to compute. But at least we can confirm that the result will be beyond $H^{n+1}(G, U(1))$.

Liang Kong, TL, Xiao-Gang Wen, Zhi-Hao Zhang, Hao Zheng, Classification of topological phases with finite internal symmetries in all dimensions, [arXiv:2003.08898].

Higher symmetry

- For a more general higher symmetry, we may replace $n\mathrm{Rep}G$ for any symmetric higher categories $\mathcal{E}_n$, $n = 0, 1, 2, \ldots$, where $\mathcal{E}_n = \Omega\mathcal{E}_{n+1}$.
- There will an integer p such that

$$\mathcal{E}_{n+1} = \Sigma\mathcal{E}_n, \forall n > p$$

$$\mathcal{E}_{p+1} \neq \Sigma\mathcal{E}_p.$$

  In other words, the $(p+1)$D defects are elementary. Higher dimensional defects are condensation descendants. Such higher symmetry is called a p-symmetry. The usual global symmetry specified by a group G is thus a 0-symmetry.

Topological phases with higher symmetry

Topological phases with higher symmetry are classified in a similar manner:

$$\mathrm{Hom}(\mathbf{1}_{n+1}, \mathbf{1}_{n+1}) = \mathcal{E}_n,$$

$$C \in \mathcal{E}_{n+1}, \mathcal{C} = \mathrm{Hom}(C, C),$$

$$\begin{array}{ccc} & \mathcal{E}_n & \\ \swarrow & & \searrow \\ Z_1(\mathcal{E}_n) & \xrightarrow{\simeq} & Z_1(\mathcal{C}) \end{array}$$

Thanks for attention!