Higher Dimensional Topological Order, n-Category and A Classification in 3+1D

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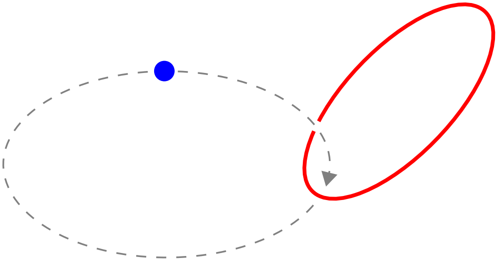
Working Definitions

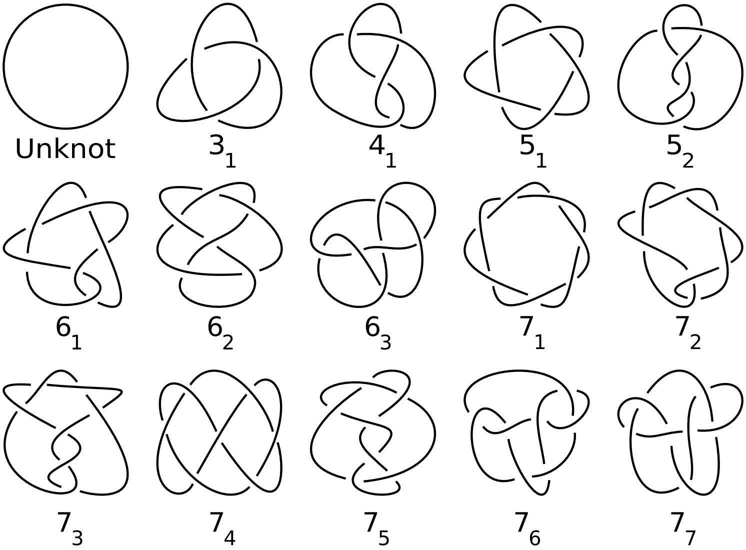
- Physical definition: topological orders are gapped quantum liquid states without any symmetry.
- In this talk we focus on topological defects and excitations.
  excitations viewed as defects between trivial defects
- Topological defects/excitations: Gapped defects. At fixed-point, physical observables depend on only their topologies (no dependence on metrics, scales, ...,)
- n+1D topological order: the collection of topological excitations in n spacial dimensions

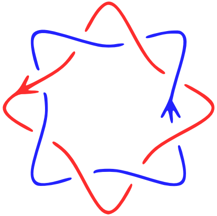
3+1D Topological Order



- String-like excitations in addition to point-like excitations.
- They can braid with each other.

Knots and Links?





Difficult!
However, for a classification we do not need to study them!

Outline

- Boundary-bulk duality: Boundary uniquely determines bulk
- Trivial mutual braiding of low-dimensional excitations
  $\Rightarrow$ point-like excitations determines a hidden "gauge group"
- Condensation of excitations with trivial statistics
  condensing enough excitations can create a boundary
- Applying above ideas in 3+1D leads to a classification:
  3+1D topological orders can all be obtained by gauging SPT.

Topological order (potentially anomalous)

n-cat describes excitations of n different dimensions:

spacial dimension of excitations	
0	particle (point-like)
1	string
2	membrane
⋮	⋮
n-1	

If the total spacial dimension is n, the excitations of top dimension n-1 can fuse but can not braid with any other excitation.

- n+1D boundary: fusion n-cat
- 2+1D boundary: particles and strings. Strings can fuse but not braid.

Topological order (anomaly-free)

Braiding is the only physical probe in topological theories.

Anomaly-free condition: Braiding non-degeneracy

All topological excitations must be detectable via braidings.

A. Kitaev, Ann. Phys. 321, 2 (2006); M. Levin, PRX 3, 021009 (2013); L. Kong and X.-G. Wen, arXiv:1405.5858

For a n-cat to be braided, the total spacial dimension needs to be at least n+1.

- (n+1)+1D bulk: non-degenerate braided fusion n-cat
- 3+1D bulk: particles and strings. They can fuse as well as braid. The braiding is non-degenerate.

Boundary-bulk duality (Holography)

Given an n+1D boundary theory, i.e., a (potentially) anomalous topological order in n+1D or a fusion n-cat,

- The boundary theory must involve at least a small neighbourhood in the bulk near the boundary.
- For topological theories there is no scale dependence, a small neighbourhood is the same as the whole bulk.

L. Kong and X.-G. Wen, arXiv:1405.5858; L. Kong, X.-G. Wen, and H. Zheng, Nucl. Phys. B 922, 62 (2017)

Boundary-bulk duality (Holography)

A boundary, a fusion n-cat, uniquely determines the bulk, a non-degenerate braided fusion n-cat,

generalized Drinfeld center

$\mathcal{Z}_n$: fusion n-cat $\to$ non-degenerate braided fusion n-cat

Concrete constructions: Turaev-Viro TQFT, Levin-Wen model, Walker-Wang model
Physically, $\mathcal{Z}_n$ calculates the anomaly of the boundary (a fusion n-cat).

Anomaly-free condition (alternative)

Has a trivial bulk if viewed as a boundary:
A fusion n-cat $\mathcal{C}$ is anomaly-free if $\mathcal{Z}_n(\mathcal{C}) = \mathbf{1}$

Low-dimensional excitations have symmetric braidings

Full braiding path between low-dimensional excitations is homotopic to trivial path.

- In 3+1D, particle and particle braid symmetrically (boson/fermion).
- In 4+1D, particle-particle and particle-string braidings are symmetric.
- In 5+1D, particle-particle, particle-string and string-string braidings are symmetric.
- . . .
- In n+1D for odd n, all the excitations of dimension $<= (n-3)/2$ braid trivially.

Point-like excitations in 3+1D or higher

They are bosons or fermions with trivial double braidings.
$\Leftrightarrow$ Point-like excitations form a symmetric fusion category
$\Leftrightarrow$ $\mathrm{Rep}(G, z)$, (G, z) is uniquely determined up to isomorphisms.
Here $z \in G$ is involutive $z^2 = 1$ and central $zg = gz, \forall g \in G$.

P. Deligne, Catégories tensorielles, Mosc. Math. J. 2 (2002), no. 2, 227-248

- $z = 1$: usual representation category $\mathrm{Rep}(G)$.
- z is nontrivial: z corresponds to the fermion number parity; the representations where z acts non-trivially are fermions.
  To emphasize the fermionic nature, for non-trivial z, we use the notations $G^f \equiv (G, z), \mathrm{sRep}(G^f) \equiv \mathrm{Rep}(G, z), Z_2^f \equiv \{1, z\}$.

Symmetric braiding is a very strong constraint.

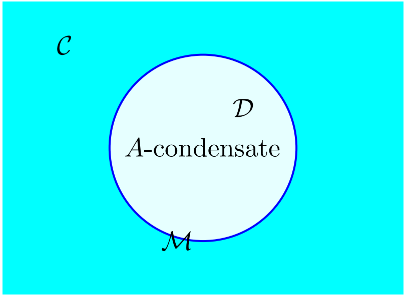
Classification in 3+1D

- In 3+1D, there are only point-like and string-like excitations.
- Point-like excitations must have trivial statistics, fully determined by (G, z).
- Braiding non-degeneracy puts very strong constraints on the string-like excitations.
  Expect: determined by (G, z) plus certain extra data
- Hard to extract due to technical difficulty on braided fusion 2-cats.
- A "detour": condensation

Conjecture: similar results for odd spacial dimensions:

- Low dimensional excitations have symmetric braidings
  $\Rightarrow$ higher representations of higher (super-)group.
- High dimensional excitations are determined by such higher group to certain extent.

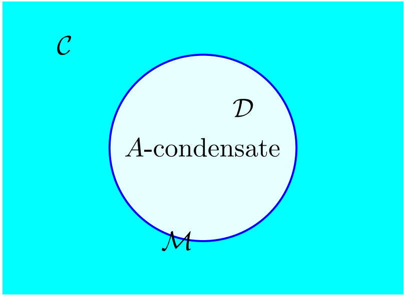
Condensation



In 2+1D, condensing A in phase $\mathcal{C}$, we obtain a new phase $\mathcal{D}$, together with a gapped defect $\mathcal{M}$ between $\mathcal{C}$ and $\mathcal{D}$. Condensed anyons A must have trivial self statistics and trivial mutual statistics among themselves.

- A condensed phase $\mathcal{D} = \mathcal{C}_A^{loc}$: local A-modules in $\mathcal{C}$ ("local" means having trivial mutual statistics with A)
- Induced gapped interface $\mathcal{M} = \mathcal{C}_A$: (all) A-modules in $\mathcal{C}$

Condensation



- When "enough" excitations are condensed (A is Langragian algebra) such that $\mathcal{D} = \mathrm{Vect}$ is the trivial phase, $\mathcal{M}$ is a boundary. By boundary-bulk duality we have $\mathcal{C} = \mathcal{Z}_1(\mathcal{M})$. However, in 2+1D not every $\mathcal{C}$ contains a Langragian algebra.
- Fortunately, in 3+1D there is always "large" enough A (the low dimensional excitations) to create a boundary, which in turn determines the bulk.
  Just need to study such boundary!

All-boson (AB) 3+1D topological orders

PRX 8, 021074 (2018), arXiv:1704.04221

In 3+1D, when all point-like excitations are bosons, they form $\mathrm{Rep}(G)$. Condense them [$A = \mathrm{Fun}(G)$]:

- New phase has no point-like excitations.
- Also no string-like excitations due to braiding non-degeneracy. Everything is confined, trivial phase.
- Obtain a boundary (fusion 2-cat) that also has no point-like excitation, only string-like excitations
- Study the braiding between the string on boundary with particles: Strings on boundary given by G.

All-boson (AB) 3+1D topological orders

PRX 8, 021074 (2018), arXiv:1704.04221

- Such fusion 2-cat classified by (G, ω_4), $\omega_4 \in H^4[G, U(1)]$, just G-graded 2-vector-spaces $2\mathrm{Vect}_G^{\omega_4}$.

  Similar as bosonic symmetric protected topological (SPT) phases

  X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G Wen, Phys. Rev. B 87, 155114 (2013), Science 338, 1604 (2012)

- Non-degenerate braided fusion 2-cat whose point-like excitations are $\mathrm{Rep}(G)$, are all of the form $\mathcal{Z}_2(2\mathrm{Vect}_G^{\omega_4})$

  Dijkgraaf-Witten gauge theory in 3+1D

  R. Dijkgraaf and E. Witten, Comm. Math. Phys. 129, 393 (1990)

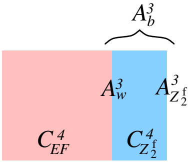
  Gauged bosonic SPT

Emergent-fermion (EF) 3+1D topological orders

PRX 9, 021005 (2019), arXiv:1801.08530

In 3+1D, when some point-like excitations are emergent fermions, they form $\mathrm{sRep}(G^f)$. Condense all bosonic point-like excitations [$A = \mathrm{Fun}(G_b = G^f/Z_2^f)$]:

- In the new phase, point-like excitations form $\mathrm{sRep}(Z_2^f) \simeq \mathrm{sVect}$.
- Such 3+1D topological order is unique. Its string-like excitations can be condensed, after which a boundary with only point-like excitations $\mathrm{sRep}(Z_2^f)$ is obtained. Strictly speaking there are also Majorana chains. Explain later. In other words, it is $\mathcal{Z}_2[\mathrm{sRep}(Z_2^f)]$.
- The gapped interphase $\mathcal{A}_w^3$ between the original phase $\mathcal{C}_{EF}^4$ and $\mathcal{C}_{Z_2^f}^4 \equiv \mathcal{Z}_2[\mathrm{sRep}(Z_2^f)]$, the new phase $\mathcal{C}_{Z_2^f}^4$ and its boundary $\mathcal{A}_{Z_2^f}^3 = \mathrm{sRep}(Z_2^f)$, form a "sandwich" boundary $\mathcal{A}_b^3$ of the original phase.



Emergent-fermion (EF) 3+1D topological orders

PRX 9, 021005 (2019), arXiv:1801.08530

- Alternatively, condensing all bosons together with some strings leads to a boundary of the original phase.
- On this boundary, only non-trivial point-like excitation is the fermion. String-like excitations similarly have group-like fusion rules. Closed strings form G_b. But when considering open strings, there is an extra Z_2^m string corresponding to Majorana chain.
  There are further two cases:

  EF1 String fusion given by $G_b \times Z_2^m$.

  Classification similar as group super-cohomology theory for fermionic SPTs.

  Z.-C. Gu and X.-G Wen, Phys. Rev. B 90, 115141 (2014)

  EF2 String fusion given by a nontrivial Z_2^m extension of G_b. This case must have emergent Majorana zero modes.

  This also has counterpart in fermionic SPTs.

  A. Kapustin and R. Thorngren, arXiv:1701.08264; Q.-R. Wang and Z.-C. Gu, arXiv:1703.10937

Emergent-fermion (EF) 3+1D topological orders

PRX 9, 021005 (2019), arXiv:1801.08530

- Non-degenerate braided fusion 2-cat whose point-like excitations are $\mathrm{sRep}(G^f)$, are all of the form $\mathcal{Z}_2(\mathcal{A})$, with $\mathcal{A}$ being one of the above two types of fusion 2-cats (called EF 2-cats). They may be reailzed by higher gauge theories or more complicated tensor network models.

C. Zhu, T. Lan, and X.-G. Wen, arXiv:1808.09394.

Gauged fermionic SPT

Main result in short

All 3+1D topological orders correspond to gauged SPTs.

Summary

- Toplogical excitations form n-category
- Anomalous (anomaly-free) topological order and (non-degenerate braided) fusion n-cat
- Boundary-bulk duality and generalized Drinfeld center
- Braiding of low-dimensional excitaions must be trivial
- Condension of topological excitations
- Classification in 3+1D gauged bosonic/fermionic SPT

Thanks for attention!