An Introduction to Category Theory

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May 23, 2024

Contents

1 Outline: physical interpretations of the abstract mathematical notions

Category theory has become an inevitable language in the study of topological phases. However, many complained that category theory is too abstract. To this end, we begin this mathematical chapter by outlining the other sections and explaining the philosophy underneath, as well as the physical interpretations in topological phases.

In Section 2, we introduce the basic language of category theory. The notions of category, object, morphism, functor and natural transformation are given. At this general level, we cannot provide specific interpretations to these notions, but it should helpful to explain some categorical philosophy here. In many examples of categories, such as the category of sets, groups, vector spaces, or topological spaces, the objects are sets with certain additional structure, and morphisms are maps preserving the additional structure. However, learning category theory with these examples in mind can be harmful. One important perspective that category theory distinguishes from set theory, is that objects are not assumed to have any internal structures like an structured set, and all the focus is on the morphisms, and higher analogs like functors and natural transformations. In other words, objects are treated like black boxes; all we learn about an object is from the morphisms, which are the relations between objects. The "internal structure" of an object, even if there seems to be, is mere an illusion from the knowledge on the morphisms. In fact, this philosophy is very much like the way we perceive the physical world: all our knowledge comes from our observations or measurements, from the interactions between "physical objects". In "reductionism" physics, we assume that "physical objects" have certain internal structures (consist of smaller and smaller particles) which greatly helped simplifying our explanation of the nature. However, such fundamental reductionism assumption can never be justified by experiments directly, and we do see signs of its breaking down in modern physics. Can we study the nature without the reductionism assumption? Indeed, the categorical philosophy is against reductionism from the very beginning, and category theory is the corresponding methodology to organize knowledge without reductionism. The power of category theory, as implied by Yoneda lemma for example, is not less than the traditional set theoretical or reductionism approach.

In Section 3, we introduce the Yoneda lemma and universal property. This section aims to further convey the categorical philosophy based on some solid results; however, it should safe for readers who are interested in physical applications to skip this section.

In Section 4, we introduce a special kind of category that is relevant in the study of topological phases, tensor category. At this stage, we can interpret the objects as point-like excitations, or (quasi-)particles. The tensor product of objects can be interpreted as viewing two excitations as one (fusion of particles). Still, as explained above, the morphisms, which stands for the physical processes (operators) of the excitations, are the only things that are observable. There is also another important assumption, "rigidity", that objects have dual objects, which physically means that particles should always have antiparticles.

In Section 5, we describe the graphical methods to compute the relations of morphisms. Physically, these graphs can be viewed as the world lines that describe the evolution processes of the particles.

In Section 6, we thoroughly introduce the theory of unitary fusion category.

They are more refined a subclass of rigid tensor category. The unitary structure is a physical assumption that corresponds to reflection positivity. The name "fusion" is short for several assumptions, which makes it possible for us to choose a finite basis of the graphical calculus, and explicitly express the physical data encoded in the abstract categorical language. Physically, a unitary fusion category describes the quasiparticles on the gapped 1+1D boundary of a 2+1D topological ordered phase. But interestingly, due to the boundary-bulk duality of topological phases, the same unitary fusion category can also be used to construct the 2+1D bulk topological order. More precisely, the graphs of the unitary fusion category, on one hand can be understood as the world lines of boundary quasiparticles, on the other hand can be understood as the ground state wavefunction of the bulk topological order.

In Section 7, we introduce the theory of module category over tensor category. The graph involving a module category can be understood as the ground state wavefunction of a gapped boundary (or a gapped domain wall for a bimodule category), and the functors between module categories correspond to excitations on the boundary. Indeed the module functors form a unitary fusion category that is Morita equivalent to the one used to construct bulk wavefunction, which established the boundary-bulk duality. This section focus on introducing concepts and pictures instead of mathematical details.

In Section 8, we compute in detail the example of module categories over $\mathrm{Rep}(\mathbb{Z}_n)$, which helps understanding the general concepts introduced in Section 7. This is the categorical treatment of the boundary theory of $\mathbb{Z}_n$ string-net model.

In Section 9, we add an additional structure "braiding" to unitary fusion category. The resulting unitary braided fusion category are now able to describe quasiparticles in 2+1D topological orders. They are also referred to as "anyon models". We also explain how unitary braided fusion categories are used to describe topological phases with symmetries.

In Section 10, we introduce the theory of algebras in tensor category. They are directly related to the physics of (self-bosonic) anyon condensation, as well as the corresponding boundary and domain wall theory. On the other hand, they are very useful in various constructions in tensor category theory and a must to know.

In Section 11, we apply the above knowledge and discuss the stacking of topological phases in terms of categorical language.

2 Some Basic Language of Category Theory

Definition 2.1 (Category)**.** A category $\mathcal{C}$ has the following data:

1. Objects $A, B, C, \cdots$. The collection of all objects is denoted by $\mathrm{Ob}(\mathcal{C})$ ($A \in \mathcal{C}$ is short for $A \in \mathrm{Ob}(\mathcal{C})$).

2. Morphisms $f, g, \cdots$ between each ordered pair (A, B), denoted by $f : A \to B$ or $A \xrightarrow{f} B$. The collection of all morphisms from A to B is denoted by

Hom(A, B).

3. For each triple (A, B, C), and morphism $f : A \to B$ and $g : B \to C$, there exists a unique morphism $gf : A \to C$, called the composition of f and g.

4. For each object A, a morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$, called the identity morphism.

They satisfy

1. Associativity: $(hg)f = h(gf)$.

2. Identity morphisms are the units of composition: $\forall f \in \mathrm{Hom}(A, B), f = \mathrm{id}_B f = f \,\mathrm{id}_A$.

Remark 1. Note that our definition of category does not rely on set theory. But for applications it is convenient to consider set theoretic models. A category $\mathcal{C}$ is called *small* if $\mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}(A, B)$ are sets. For small categories, we can safely use set theoretic structures such as Cartesian product, subset, function or map, and so on. It worth noting that the category of sets, **Set**, whose objects are sets and morphisms are functions, is not small, because the collection of all sets is no longer a set. However, unless explicitly specified, in this book we only deal with small categories.

A morphism $f \in \mathrm{Hom}(A, B)$ is called an isomorphism if it is invertible, i.e. $\exists g \in \mathrm{Hom}(B, A)$ such that $fg = \mathrm{id}_B$, $gf = \mathrm{id}_A$, and g is denoted by f^{-1}. Two objects A, B are called isomorphic, denoted by $A \cong B$, if there exists an isomorphism between them.

Sometimes we will consider the subcategory of a given category. Note that there are two levels of structure; we can take subsets of both objects and morphisms. To be precise, by a *full* subcategory $\mathcal{B}$ of $\mathcal{C}$, we mean taking a subset of objects $\mathrm{Ob}(\mathcal{B}) \subset \mathrm{Ob}(\mathcal{C})$ but all the morphisms $\mathrm{Hom}_{\mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Definition 2.2 (Functor)**.** A functor F between two categories $\mathcal{C}, \mathcal{D}$ is given by

1. An object map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & \to & \mathrm{Ob}(\mathcal{D}) \\ A & \mapsto & F(A), \end{array}$$

2. Morphism maps

$$\begin{array}{rcl} \mathrm{Hom}(A, B) & \to & \mathrm{Hom}(F(A), F(B)) \\ f & \mapsto & F(f), \end{array}$$

satisfying

1. $F(gf) = F(g)F(f)$.

2. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

There is an identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ for any category $\mathcal{C}$, whose object map and morphism maps are both identity $\mathrm{id}_{\mathcal{C}}(A) = A, \mathrm{id}_{\mathcal{C}}(f) = f$.

Definition 2.3 (Natural Transformation)**.** A natural transformation $\nu : F \Rightarrow G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is given by a collection of morphisms $\nu_A \in \mathrm{Hom}(F(A), G(A))$ for each object $A \in \mathcal{C}$ such that $\forall f \in \mathrm{Hom}(A, B)$

$$G(f)\nu_A = \nu_B F(f). \tag{1}$$

In category theory, such condition is more commonly expressed in terms of the commutative digram

$$\begin{array}{ccc} F(A) & \xrightarrow{\nu_A} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\nu_B]{} & G(B) \end{array} \quad . \tag{2}$$

We will call such diagram as the natural square.

All the functors between two categories $\mathcal{C}, \mathcal{D}$ form a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, whose objects are functors and morphisms are natural transformations. The composition of natural transformations ν, τ is given by $(\tau\nu)_A = \tau_A \nu_A$. The morphisms ν_A of the natural transformation ν are also called functorial. If ν is an isomorphism in the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, then all its components ν_A are isomorphisms. In this case ν is called a natural isomorphism, and by abuse of notation, its components ν_A are also called natural isomorphisms.

One may define two categories to be isomorphic by the existence of invertible functors between them. However, this notion is too strict and not practical in many applications. The reason behind is the "higher morphisms" – natural transformations – between functors. We instead need the following weaker notion:

Definition 2.4 (Equivalence of category)**.** Two categories $\mathcal{C}, \mathcal{D}$ are called equivalent ($\mathcal{C} \simeq \mathcal{D}$) if there exist two functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathrm{id}_{\mathcal{C}}, FG \cong \mathrm{id}_{\mathcal{D}}$ as objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}), \mathrm{Fun}(\mathcal{D}, \mathcal{D})$.

3 Yoneda Lemma and Universal Property

This is an optional section for interested readers. In this section we introduce some basic constructions and results in category theory. For the moment, the contents in this section are not directly relevant to physical applications. However, this section illustrates the philosophy of category theory; the authors believe that the "categorical way of thinking" is beneficial for understanding the following sections.

3.1 Yoneda Lemma

Think about a fundamental question: In real world, how do we learn about something new? As the first step, we give the new thing a name, no matter meaningful or meaningless. Without loss of generality, let's say the name is "A". Next, we may use everything that we already have but we may or may not know very well, to test A, or interact with A. In reality, such process is nothing but experiment. Finally, our knowledge of A is concluded form these experiments, which in turn give the name A its meaning.

Categorically, the above thinking says that the abstract object A is in fact defined by $\mathrm{Hom}(A,-)$, where $-$ stands for an arbitrary object (everything we have) and $\mathrm{Hom}(A,-)$ is how A interacts with everything we have (all the information we can learn from our experiments). Yoneda lemma and Yoneda embedding are just the rigorous and precise statements of the above simple thinking.

Given a category $\mathcal{C}$ and an object $A \in \mathcal{C}$, $\mathrm{Hom}(A,-)$ is in fact a functor from $\mathcal{C}$ to **Set**, the category of sets, whose objects are sets and morphisms are maps between sets. The object and morphism maps of $\mathrm{Hom}(A,-)$ are

$$\begin{aligned} X \in \mathcal{C} &\mapsto \mathrm{Hom}(A,X) \in \mathbf{Set}, \\ (f : X \to Y) &\mapsto (\mathrm{Hom}(A,f) : \mathrm{Hom}(A,X) \to \mathrm{Hom}(A,Y)), \end{aligned}$$

where $\mathrm{Hom}(A,f)$ is naturally given by composition from the left (push-forward)

$$\begin{aligned} \mathrm{Hom}(A,f) : \mathrm{Hom}(A,X) &\to \mathrm{Hom}(A,Y), \\ (x : A \to X) &\mapsto (fx : A \to X \to Y). \end{aligned}$$

$\mathrm{Hom}(A,-)$ is called the (co-variant) hom-functor. Dually, we can define a *contra-variant* hom-functor $\mathrm{Hom}(-,A)$:

$$\begin{aligned} X &\mapsto \mathrm{Hom}(X,A), \\ (f : X \to Y) &\mapsto \begin{pmatrix} \mathrm{Hom}(f,A) : \mathrm{Hom}(Y,A) & \to & \mathrm{Hom}(X,A) \\ (y : Y \to A) & \mapsto & (yf : X \to Y \to A) \end{pmatrix}. \end{aligned}$$

Note that the morphism map reverses the direction of the arrow, which is why the functor is called contra-variant. It is not a difficult observation, that for any categorical statement, one can reverse the directions of all arrows and obtain a dual statement. The dual statement has the same authenticity as the original one. Formally, for any category $\mathcal{C}$, we can define a dual category $\mathcal{C}^{\mathrm{op}}$ whose objects are the same as $\mathcal{C}$ while all morphisms are reversed $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(B,A)$. Thus, a contra-variant functor from $\mathcal{C}$ to $\mathcal{D}$ is nothing but a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$.

We are now ready to state the Yoneda lemma,

Lemma 3.1 (Yoneda lemma)**.** For a category $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathbf{Set}$, and an object $A \in \mathcal{C}$, let $\mathrm{Nat}(\mathrm{Hom}(A,-),F)$ be the set of natural transformations from $\mathrm{Hom}(A,-)$ to F. $\mathrm{Nat}(\mathrm{Hom}(A,-),F)$ is in bijection with $F(A)$. Dually, for a functor $G : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, $\mathrm{Nat}(\mathrm{Hom}(-,A),G)$ is in bijection with $G(A)$.

Proof. Let $\nu : \mathrm{Hom}(A, -) \Rightarrow F$ be a natural transformation and $f : A \to X$ a morphism. Consider the natural square

$$\begin{array}{ccc} \mathrm{Hom}(A,A) & \xrightarrow{\nu_A} & F(A) \\ \downarrow{\scriptstyle \mathrm{Hom}(A,f)} & & \downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}(A,X) & \xrightarrow{\nu_X} & F(X) \end{array} \tag{3}$$

Pick a special element $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ and check its image

$$\begin{array}{ccc} \mathrm{id}_A \in \mathrm{Hom}(A,A) & \xmapsto{\nu_A} & u \equiv \nu_A(\mathrm{id}_A) \in F(A) \\ \downarrow{\scriptstyle \mathrm{Hom}(A,f)} & & \downarrow{\scriptstyle F(f)} \\ f\,\mathrm{id}_A = f \in \mathrm{Hom}(A,X) & \xmapsto{\nu_X} & \nu_X(f) = F(f)(u) \in F(X) \end{array} \tag{4}$$

We see that ν is uniquely determined by the image of id_A under ν_A. Therefore, the bijection between $\mathrm{Nat}(\mathrm{Hom}(A, -), F)$ and $F(A)$ is given by

$$\begin{aligned} \mathrm{Nat}(\mathrm{Hom}(A,-),F) &\rightleftharpoons F(A), \\ \nu &\mapsto \nu_A(\mathrm{id}_A), \\ \nu_X(f) = F(f)(u) &\leftarrow\!\shortmid u. \end{aligned}$$

The dual case is similarly proven. □

Take $G = \mathrm{Hom}(-, B)$ in the above, we see that

$$\mathrm{Nat}(\mathrm{Hom}(-,A), \mathrm{Hom}(-,B)) \cong \mathrm{Hom}(A,B). \tag{5}$$

In fact, given a morphism $f : A \to B$, it uniquely determines a natural transformation from $\mathrm{Hom}(-, A)$ to $\mathrm{Hom}(-, B)$ by composition from the left. Thus, the contra-variant hom-functor defines a functor from $\mathcal{C}$ to the category of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ which is *fully faithful*, namely the morphism maps are bijections. In other words, $\mathcal{C}$ can be viewed as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$.

Corollary 3.2 (Yoneda embedding)**.** Denote contra-variant hom-functor by

$$\begin{aligned} Y : \mathcal{C} &\to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}), \\ A &\mapsto Y(A) \equiv \mathrm{Hom}(-,A), \\ (f : A \to B) &\mapsto Y(f) = \left\{ \begin{array}{rcl} Y(f)_X : \mathrm{Hom}(X,A) & \to & \mathrm{Hom}(X,B) \\ (g : X \to A) & \mapsto & (fg : X \to A \to B) \end{array} \right\} \end{aligned}$$

Y is fully faithful. Dually, the co-variant hom-functor also gives a fully faithful functor $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$.

Exercise 3.1. Check that the morphism map of the Yoneda embedding is indeed the bijection in the Yoneda lemma.

Exercise 3.2. The bijection $\mathrm{Nat}(\mathrm{Hom}(A,-),F) \cong F(A)$ in the Yoneda Lemma is further "natural" in A and F, by viewing $\mathrm{Nat}(\mathrm{Hom}(A,-),F)$ and $F(A)$ as two functors from $\mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ to $\mathbf{Set}$:

1. The object maps are just $\mathrm{Nat}(\mathrm{Hom}(A,-),F)$ and $F(A)$. Figure out the morphism maps.

2. Write down the corresponding natural squares and prove that they commute.

In a less formal but more philosophical sentence, the Yoneda embedding means that all the information of an abstract category $\mathcal{C}$ is encoded in the concrete hom-sets $\mathrm{Hom}(A,B)$ for arbitrary $A, B \in \mathcal{C}$. An object A is concretely realized as $\mathrm{Hom}(-,A)$ or $\mathrm{Hom}(A,-)$, namely all it relations to the others; an abstract morphism $f : A \to B$ is concretely realized as a collection of maps from $\mathrm{Hom}(-,A)$ to $\mathrm{Hom}(-,B)$ or from $\mathrm{Hom}(B,-)$ to $\mathrm{Hom}(A,-)$, namely how relations to A are carried over to relations to B.

We conclude the philosophy reflected in this section in the following:

The relation (morphism) is all.

3.2 Universal Property

Recall that in set theory, we often specify a subset of U by certain property Q of the elements

$$\{x \in U | x \text{ satisfies } Q\}.$$

Though usually not mentioned explicitly, the above means the subset of *all* elements satisfying Q; in other words, the above means the *largest* subset whose elements satisfy Q.

In category theory, we also need to specify the *extreme* (largest, smallest or in other appropriate senses) notion satisfying certain properties. The corresponding formulation is known as *universal property*, which is probably the most important notion in category theory. Almost every good mathematical notion, such as product, sum, center and so on, can be reformulated in category theory as certain type of universal property.

We begin with the simplest case

Definition 3.1. An object T in $\mathcal{C}$ is called *terminal object* if for any object A, there exists a unique morphism from A to T, namely $\mathrm{Hom}(A,T)$ has exactly one element. Dually, an object S in $\mathcal{C}$ is called *initial object* if for any object A, there exists a unique morphism from S to A.

Terminal and initial objects may not exist; if exist, they are unique up to unique isomorphisms.

Example 3.1. In the category of sets, $\mathbf{Set}$, the initial object is the empty set $\emptyset$ (the unique map $\emptyset \to A$ to any set A is the empty map). Any set with only one element is a terminal object.

We now give the most general formulation of universal property. Say the notion we want to define is in some working category $\mathcal{C}$. To specify the property we are interested in, we usually consider some diagrams of morphisms in $\mathcal{C}$. Note that a diagram is nothing but some indexed objects and morphisms put together in some organized way. Formally, we describe a diagram by a functor $F : \mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is an indexing category whose objects and morphisms are organized the same way as the diagram, and the functor F picks out the indexed objects and morphisms in $\mathcal{C}$. Then the universal property can be thought of as the part of another diagram $G : \mathcal{K} \to \mathcal{C}$ that is closest to or farthest from $F : \mathcal{J} \to \mathcal{C}$. Rigorously, we take the terminal or initial object in the following auxiliary category

Definition 3.2. Given two functors $F : \mathcal{J} \to \mathcal{C}$ and $G : \mathcal{K} \to \mathcal{C}$, the *comma category* $(F \downarrow G)$ is as the following

- Objects are triples $(j \in \mathcal{J}, k \in \mathcal{K}, \alpha : F(j) \to G(k))$.
- Morphisms from (j_1, k_1, α_1) to (j_2, k_2, α_2) are pairs $(f : j_1 \to j_2, g : k_1 \to k_2)$ such that the following diagram commutes.

$$\begin{array}{ccc} F(j_1) & \xrightarrow{\alpha_1} & G(k_1) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(g)} \\ F(j_2) & \xrightarrow{\alpha_2} & G(k_2) \end{array} \tag{6}$$

  The composition of morphism is pairwise.

Definition 3.3 (Universal property)**.** A notion is of universal property if it is defined as, or corresponds to the terminal or initial object in appropriate comma category.

Example 3.2. Let $\varnothing$ denote the empty category. There is a unique functor $\varnothing \to \mathcal{C}$, namely the empty functor. We have $\mathcal{C} = (\mathrm{id}_{\mathcal{C}} \downarrow \varnothing \to \mathcal{C}) = (\varnothing \to \mathcal{C} \downarrow \mathrm{id}_{\mathcal{C}})$. Thus terminal and initial objects themselves are universal.

Example 3.3. Denote by $*$ the category with only one object $*$ and only identity morphism id_*. An object x of $\mathcal{C}$ can be identified with a functor $x : * \to \mathcal{C}$ which maps $*$ to x and id_* to id_x. For a diagram $F : \mathcal{J} \to \mathcal{C}$, consider the terminal object in $(F \downarrow x)$, which is $(c, *, \alpha)$. For any object $(i, *, \beta)$ in $(F \downarrow x)$ there exists a unique morphism $\bar{\beta} : i \to c$ such that $\beta = \alpha F(\bar{\beta})$

$$\begin{array}{ccc} F(c) & \xrightarrow{\alpha} & x \\ {\scriptstyle \exists! F(\bar{\beta})}\uparrow & \nearrow{\scriptstyle \beta} & \\ F(i) & & \end{array} \tag{7}$$

We see that $F(c)$ is the "closest" to x in diagram F. The morphism $\alpha : F(c) \to x$ is called a *universal morphism*. Any morphism $\beta : F(i) \to x$ factors through α.

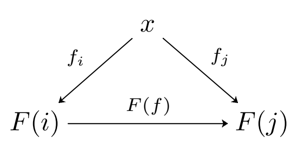
Dually, the initial object in $(x \downarrow F)$ also gives a universal morphism that any morphism $x \to F(i)$ factors through.

In particular, the identity morphism id_x is universal; it is both the terminal object in $(\mathrm{id}_{\mathcal{C}} \downarrow x)$ and the initial object in $(x \downarrow \mathrm{id}_{\mathcal{C}})$. The above is also true for any isomorphism $x \cong y$.

Another widely used application of universal property is the limit and colimit of certain diagram $F : \mathcal{J} \to \mathcal{C}$. For this purpose we like to work in the category of diagrams of shape $\mathcal{J}$, namely $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$. F is an object in $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$, and by abuse of notation we write $F : * \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ as in Example 3.3. Denote by $\Delta(x) : \mathcal{J} \to \mathcal{C}$ the constant functor that maps all objects in $\mathcal{J}$ to $x \in \mathcal{C}$ and all morphisms in $\mathcal{J}$ to id_x. Intuitively, one can think that $\Delta(x)$ shrinks the diagrams of shape $\mathcal{J}$ to a single point x. A morphism $f : x \to y$ gives a natural transformation $\Delta(f) : \Delta(x) \Rightarrow \Delta(y)$ by $\Delta(f)_i = f$, $\forall i \in \mathcal{J}$. We then have the diagonal functor $\Delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ which maps object x to constant functor $\Delta(x)$ and maps morphism f to $\Delta(f)$.

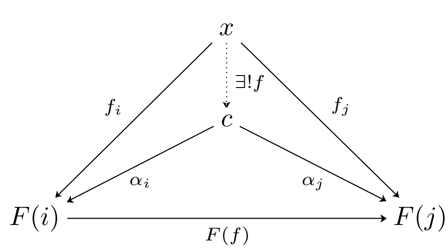
Definition 3.4 (Limit and colimit)**.** The limit of a diagram $F : \mathcal{J} \to \mathcal{C}$ is the terminal object in the comma category $(\Delta \downarrow F)$. Dually, the colimit of F is the initial object in the comma category $(F \downarrow \Delta)$.

Let's unpack the above definition of limit. An object in $(\Delta \downarrow F)$ is in fact an object $x \in \mathcal{C}$ with a natural transformation $\Delta(x) \Rightarrow F$, which amounts to a collection of morphisms $f_i : x \to F(i)$, $i \in \mathcal{J}$ satisfying the following commuting diagram



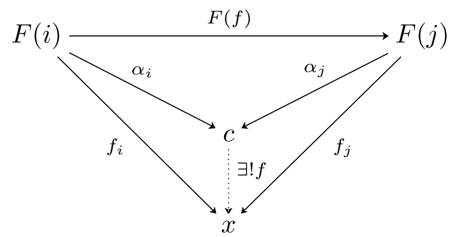
(8)

If we imagine that $F : \mathcal{J} \to \mathcal{C}$ is a diagram in the plane and x is a point above the plane, (8) has the shape of a cone. Thus, $(\Delta \downarrow F)$ is also called the category of cones over F. The limit of F is a universal cone $\alpha_i : c \to F(i)$ that any cone factors through



(9)

Dually, the colimit of F is a universal cocone $\alpha_i : F(i) \to c$ that any cocone factors through



(10)

Example 3.4. For an empty diagram, namely $\mathcal{J} = \varnothing$ the empty category and $F : \varnothing \to \mathcal{C}$ the empty functor, we have $(\Delta \downarrow F) = \mathcal{C} = (F \downarrow \Delta)$. Thus, the limit of empty diagram $\varnothing \to \mathcal{C}$ is the terminal object. The colimit of empty diagram $\varnothing \to \mathcal{C}$ is the initial object.

Alternatively, initial and terminal objects can be characterized by the following:

Example 3.5. The limit of id_C is the initial object and the colimit of id_C is the terminal object.

It is convenient to name the (co-)limit of diagrams of some special shape $\mathcal{J}$:

Definition 3.5. Let $\mathcal{J}$ be a discrete category, i.e., there are no other morphisms than the identity ones. The (co-)limit of $F : \mathcal{J} \to \mathcal{C}$ is called the *(co-)product*.

Definition 3.6. Let $\mathcal{J} = \bullet \rightrightarrows \bullet$, i.e., a category with two objects and two parallel morphisms besides the identity ones. The (co-)limit of $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is called the *(co-)equalizer*.

Definition 3.7. Let $\mathcal{J} = \bullet \to \bullet \leftarrow \bullet$. The limit of $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is called the pullback. Dually, for $\mathcal{J} = \bullet \leftarrow \bullet \to \bullet$, the colimit of $\mathcal{F} : \mathcal{J} \to \mathcal{C}$ is called the pushout.

Remark 2. The category of categories, **Cat**, has categories as objects and functors as morphisms. However, **Cat** is in fact a (strict) *2-category*, since there are 2-morphisms between functors, namely the natural transformations. In a 2-category there is similarly a notion of 2-limit. The only difference from limit in category is that for every commuting diagram one needs to specify a 2-morphism: the diagram commutes up to the specified 2-morphism. In this sense, the notion of comma category defined in this section, is the 2-pullback of the diagram $\mathcal{J} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{K}$, thus of universal property. We leave it for the interested readers as a long term exercise that almost every good construction in mathematics is universal.

It is important to know if certain limit exists or not in a given category. We give a theorem about this issue without proof.

Definition 3.8. A category $\mathcal{C}$ is called (finitely) (co-)complete if the (co-)limit of any (finite) diagram $F : \mathcal{J} \to \mathcal{C}$ exists.

Theorem 3.3. A category $\mathcal{C}$ is (finitely) (co-)complete if all (finite) (co-)products and (co-)equalizers exist in $\mathcal{C}$.

Example 3.6. In the category of sets, **Set**, the product is just the Cartesian product. The equalizer of two maps $f, g : A \to B$ is the subset $\{x \in A | f(x) = g(x)\}$ with the inclusion map. The coproduct is the disjoint union. The coequalizer is the quotient set. Therefore, **Set** is complete and cocomplete.

Example 3.7. In the category of finite dimensional vector spaces, **Vec**, the finite product and finite coproduct coincide, which is the direct sum. Given two linear maps $f, g : A \to B$, their equalizer is $\ker(f - g)$ and coequalizer is $B/\mathrm{im}(f - g)$. Therefore, **Vec** is finitely complete and finitely cocomplete.

4 Rigid Tensor Category

Let $\mathcal{C}$ be a category. $\mathcal{C} \times \mathcal{C}$ is also a category, whose objects are $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$ and morphisms are $\mathrm{Hom}((A, B), (C, D)) = \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, D)$, and $\mathrm{id}_{(A,B)} = (\mathrm{id}_A, \mathrm{id}_B), (f, f')(g, g') = (fg, f'g')$.

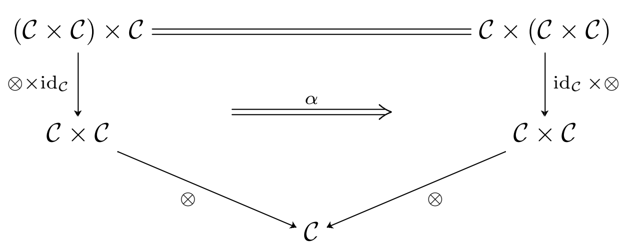
Definition 4.1 (Tensor Category)**.** A tensor category (or monoidal category) is a category $\mathcal{C}$ equipped with

1. A $\otimes$ functor

$$\begin{aligned} \otimes : \mathcal{C} \times \mathcal{C} &\to \mathcal{C} \\ (A, B) &\mapsto A \otimes B \\ (f, g) &\mapsto f \otimes g, \end{aligned}$$

2. Associator: a natural isomorphism

$$\alpha : \otimes(\otimes \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \otimes(\mathrm{id}_{\mathcal{C}} \times \otimes),$$



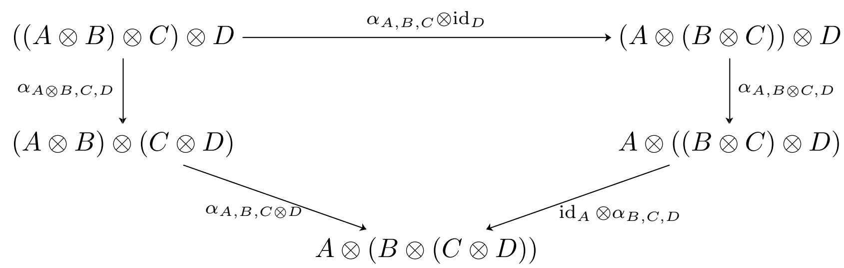
which consists of $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for $A, B, C \in \mathcal{C}$,

3. A unit object $\mathbf{1} \in \mathrm{Ob}(\mathcal{C})$, together with two natural isomorphisms

$$\lambda : \mathbf{1} \otimes - \Rightarrow \mathrm{id}_{\mathcal{C}}, \quad \rho : - \otimes \mathbf{1} \Rightarrow \mathrm{id}_{\mathcal{C}},$$

(Here the functor $\mathbf{1} \otimes -$ is understood as $\mathbf{1} \otimes -(A) = \mathbf{1} \otimes A$, $\mathbf{1} \otimes -(f) = \mathrm{id}_{\mathbf{1}} \otimes f$, and similar for $- \otimes \mathbf{1}$) whose components are $\lambda_A : \mathbf{1} \otimes A \to A$, $\rho_A : A \otimes \mathbf{1} \to A$ for each $A \in \mathcal{C}$, satisfying

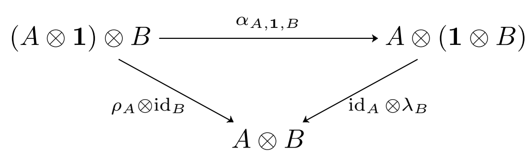
1. Pentagon equations: $\forall A, B, C, D \in \mathcal{C}$ the diagram



(11)

   commutes.

2. Triangle equations: $\forall A, B \in \mathcal{C}$ the diagram



(12)

   commutes.

Exercise 4.1. Try to *unpack* Definition 4.1 with the definitions of functor and natural isomorphisms.

Answer:

1. $\otimes$ is a functor, thus

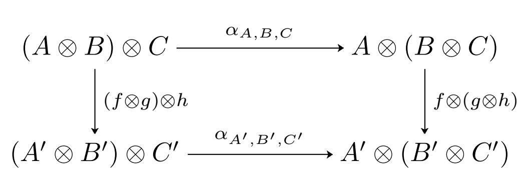
$$\otimes((f', g')(f, g)) = \otimes((f', g')) \otimes ((f, g)), \quad \otimes(\mathrm{id}_{(A,B)}) = \mathrm{id}_{\otimes((A,B))} . \tag{13}$$

   Namely,

$$f'f \otimes g'g = (f' \otimes g')(f \otimes g), \quad \mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A \otimes B} . \tag{14}$$

2. α is a natural isomorphism, thus

   - $\alpha_{A,B,C}$ are isomorphisms, i.e., invertible.
   - For any $f \in \mathrm{Hom}(A, A'), g \in \mathrm{Hom}(B, B'), h \in \mathrm{Hom}(C, C')$ the diagram



(15)

     commutes.

3. λ, ρ are natural isomorphisms, thus

    - λ_A, ρ_A are isomorphisms, i.e., invertible.
    - For any $f \in \mathrm{Hom}(A, A')$ the diagrams

$$\begin{array}{ccc} \mathbf{1} \otimes A & \xrightarrow{\lambda_A} & A \\ \Big\downarrow{\scriptstyle \mathrm{id}_{\mathbf{1}} \otimes f} & & \Big\downarrow{\scriptstyle f} \\ \mathbf{1} \otimes A' & \xrightarrow{\lambda_{A'}} & A' \end{array} \tag{16}$$

$$\begin{array}{ccc} A \otimes \mathbf{1} & \xrightarrow{\rho_A} & A \\ \Big\downarrow{\scriptstyle f \otimes \mathrm{id}_{\mathbf{1}}} & & \Big\downarrow{\scriptstyle f} \\ A' \otimes \mathbf{1} & \xrightarrow{\rho_{A'}} & A' \end{array} \tag{17}$$

    commute.

Remark 3. Many statements in category theory are made in a similar compact way. Remember not to overlook them and unpack like the above if necessary.

Exercise 4.2. For convenience, let us denote the commutative diagrams (11) by $\mathrm{Pen}_{A,B,C,D}$ and (12) by $\mathrm{Tri}_{A,\mathbf{1},B}$. The edge morphisms in $\mathrm{Pen}_{-,-,-,-}$ and $\mathrm{Tri}_{-,-,-}$ are constructed with associativity and unit isomorphisms. Draw $\mathrm{Tri}_{\mathbf{1},A,B}$ and $\mathrm{Tri}_{A,B,\mathbf{1}}$. Show that they also commute.

Tips: Take $\mathrm{Tri}_{A,B,\mathbf{1}}$ as an example:

$$\begin{array}{ccc} (A \otimes B) \otimes \mathbf{1} & \xrightarrow{\alpha_{A,B,\mathbf{1}}} & A \otimes (B \otimes \mathbf{1}) \\ & {\scriptstyle \rho_{A\otimes B}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \rho_B} & \\ & A \otimes B & \end{array} \tag{18}$$

1. Draw $\mathrm{Pen}_{A,B,\mathbf{1},\mathbf{1}}$.

2. Put $(A \otimes B) \otimes \mathbf{1}$ and $A \otimes (B \otimes \mathbf{1})$ inside the pentagon.

3. Connect objects with appropriate morphisms. You will find that the pentagon consists of $\mathrm{Tri}_{A\otimes B,\mathbf{1},\mathbf{1}}$, $\mathrm{Tri}_{A,B,\mathbf{1}} \otimes \mathbf{1}$, $A \otimes \mathrm{Tri}_{B,\mathbf{1},\mathbf{1}}$, and two natural squares of α.

4. The commutativity of $\mathrm{Tri}_{A,B,\mathbf{1}} \otimes \mathbf{1}$ is then implied by the commutativity of $\mathrm{Pen}_{A,B,\mathbf{1},\mathbf{1}}$, $\mathrm{Tri}_{A\otimes B,\mathbf{1},\mathbf{1}}$, $\mathrm{Tri}_{B,\mathbf{1},\mathbf{1}}$, and the natural squares.

5. $\mathrm{Tri}_{A,B,\mathbf{1}}$ can be built from $\mathrm{Tri}_{A,B,\mathbf{1}} \otimes \mathbf{1}$ and three natural squares of ρ.

Definition 4.2 (Rigidity)**.** A left dual to object $A \in \mathcal{C}$ is a triple (A^*, b_A, e_A), an object A^* with two morphisms $b_A : \mathbf{1} \to A \otimes A^*, e_A : A^* \otimes A \to \mathbf{1}$ such that the diagrams

$$
\begin{array}{ccc}
(A \otimes A^*) \otimes A & \xrightarrow{\alpha_{A,A^*,A}} & A \otimes (A^* \otimes A) \\
{\scriptstyle b_A \otimes \mathrm{id}_A} \uparrow & & \downarrow {\scriptstyle \mathrm{id}_A \otimes e_A} \\
\mathbf{1} \otimes A & & A \otimes \mathbf{1} \\
{\scriptstyle \lambda_A^{-1}} \uparrow & & \downarrow {\scriptstyle \rho_A} \\
A & \xrightarrow{\mathrm{id}_A} & A
\end{array}
\qquad
\begin{array}{ccc}
A^* \otimes (A \otimes A^*) & \xrightarrow{\alpha^{-1}_{A^*,A,A^*}} & (A^* \otimes A) \otimes A^* \\
{\scriptstyle \mathrm{id}_{A^*} \otimes b_A} \uparrow & & \downarrow {\scriptstyle e_A \otimes \mathrm{id}_{A^*}} \\
A^* \otimes \mathbf{1} & & \mathbf{1} \otimes A^* \\
{\scriptstyle \rho^{-1}_{A^*}} \uparrow & & \downarrow {\scriptstyle \lambda_{A^*}} \\
A^* & \xrightarrow{\mathrm{id}_{A^*}} & A^*
\end{array}
\tag{19}
$$

commute. A right dual $({}^*A, b'_A, e'_A)$ is similarly defined. Note that (A, b_A, e_A) is right dual to A^*. Dual object is unique up to isomorphism. Tensor category $\mathcal{C}$ is called rigid if every object in $\mathcal{C}$ has left and right duals.

The diagrams (19) seems a bit complicated, however, the nature of this definition is very clear via the tricks of graphical calculus of tensor categories. The main idea is to express morphisms with string graphs, instead of commutative diagrams. And even more important, graphical calculus is, more or less, the picture of string-net model. The conventions of graphical calculus vary from people to people and we need to fix them. The following listed conventions can be used for general tensor categories. Later this chapter, we will introduce a better-looking convention that makes use of the nice properties of unitary fusion categories, which agrees with the convention in most string-net model literature. However, although that convention looks better, there are tricky ambiguity arising for special unitary fusion categories.

5 Graphical Calculus for Tensor Categories

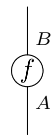
First, we fix our convention for graphical calculus:

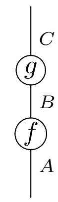
1. Identity morphisms are drawn as vertical lines with labels. In particular, $\mathrm{id}_{\mathbf{1}}$ is drawn as a dashed line, or simply omitted.

$$
\Bigg|_{A} = \mathrm{id}_A, \qquad \vdots = \mathrm{id}_{\mathbf{1}} . \tag{20}
$$

2. Compose morphisms from bottom to top. For example, say $f \in \mathrm{Hom}(A, B), g \in$

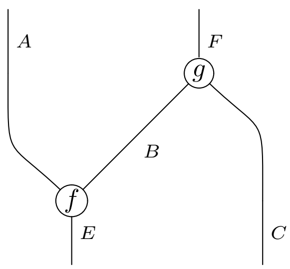
Hom(B, C),



$= \mathrm{id}_B\, f\, \mathrm{id}_A = f,$


$= gf.$ (21)

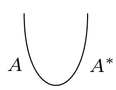
3. Take tensor product $\otimes$ for juxtaposed morphisms. Because of the coherence axioms (pentagon and triangle equations), the unit $\mathbf{1}$ and the canonical isomorphisms α, λ, ρ are implicit in graphical calculus. For example

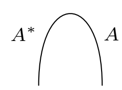


$= (\mathrm{id}_A \otimes g)\alpha_{A,B,C}(f \otimes \mathrm{id}_C) \in \mathrm{Hom}(E \otimes C, A \otimes F).$

(22)

Next, we show the power of graphical calculus by studying some properties of dual objects. We use the following graphs for b_A, e_A in Definition 4.2

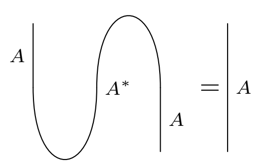


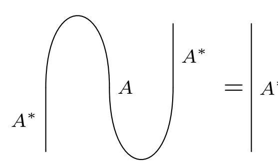
$= b_A \in \mathrm{Hom}(\mathbf{1}, A \otimes A^*),$


$= e_A \in \mathrm{Hom}(A^* \otimes A, \mathbf{1}).$

(23)

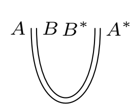
Now we can rewrite the diagrams (19) in terms of graphs, i.e.

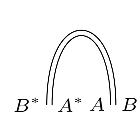


,


. (24)

Thus, rigidity simply means that strings can be bend over. With this trick one can easily find $B^* \otimes A^*$ with

$b_{A\otimes B} =$


, $e_{A\otimes B} =$


. (25)

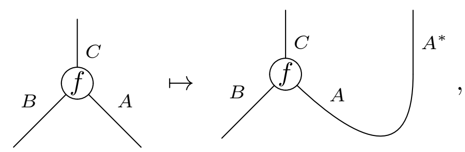
is left dual to $A \otimes B$.

Theorem 5.1. Let (A^*, b_A, e_A) be a left dual to A. $\forall B, C \in \mathcal{C}$, the following Hom-sets are isomorphic

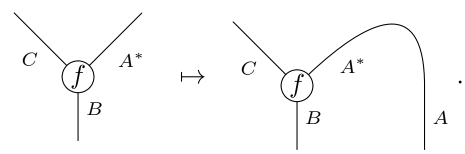
$$\mathrm{Hom}(B \otimes A, C) \cong \mathrm{Hom}(B, C \otimes A^*),$$
$$\mathrm{Hom}(B, A \otimes C) \cong \mathrm{Hom}(A^* \otimes B, C).$$

Proof. The first isomorphism maps

$$\mathrm{Hom}(B \otimes A, C) \quad \to \quad \mathrm{Hom}(B, C \otimes A^*)$$



$$\mathrm{Hom}(B, C \otimes A^*) \quad \to \quad \mathrm{Hom}(B \otimes A, C)$$



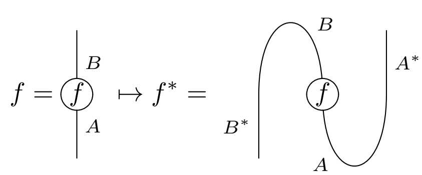
It is easy to check the two maps cancel each other. The second isomorphism maps are similar. Also, we have similar theorems for right duals. $\square$

Corollary 5.2. We have

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(\mathbf{1}, B \otimes A^*) \cong \mathrm{Hom}(B^* \otimes A, \mathbf{1}) \cong \mathrm{Hom}(B^*, A^*).$$

The following map $*$ is an isomorphism

$$* : \mathrm{Hom}(A, B) \to \mathrm{Hom}(B^*, A^*)$$



For a rigid category $\mathcal{C}$, $*$ further extends to a monoidal equivalence functor $\mathcal{C}^{\mathrm{rev}} \to \mathcal{C}^{\mathrm{op}}$, where $\mathcal{C}^{\mathrm{rev}}$ denote the same category as $\mathcal{C}$ but with reversed tensor product, $A \otimes^{\mathrm{rev}} B := B \otimes A$.

6 Unitary Fusion Category

From now on we will assume that the category $\mathcal{C}$ is a $\mathbb{C}$-linear category, i.e. Hom-sets of $\mathcal{C}$ are $\mathbb{C}$-vector spaces and composition maps of $\mathcal{C}$ are $\mathbb{C}$-bilinear; the $\otimes$ is also $\mathbb{C}$-bilinear if $\mathcal{C}$ is a tensor category. We have the following "direct sum" construction which generalizes direct sum of vector spaces in terms of more general categorical language. It may be the most useful structure for a physicist, since it allows us to choose bases and express the abstract categorical notions in terms of matrices and tensors.

Definition 6.1 (Direct sum)**.** An object $A \in \mathcal{C}$ is the direct sum of n objects $A_1, A_2, ..., A_n \in \mathcal{C}$ if there exist $2n$ morphisms $p_a \in \mathrm{Hom}(A, A_a)$, $q_a \in \mathrm{Hom}(A_a, A)$, $a = 1, \cdots, n$, satisfying

$$p_a q_b = \delta_{ab}\, \mathrm{id}_{A_a}, \quad \sum_{a=1}^{n} q_a p_a = \mathrm{id}_A\,. \tag{26}$$

We denote the direct sum by $A = \oplus_{a=1}^{n} A_a$, and the p_a, q_a morphisms are called (direct sum) decomposition morphisms, or more precisely, p_a as projection morphisms and q_a as embedding morphisms. One can check that the direct sum is both the product and the coproduct, thus it may also be called a biproduct. As limit and colimit, direct sum is unique up to unique isomorphism.

Each $f \in \mathrm{Hom}(A, B)$ uniquely determines n morphisms

$$f_a = f q_a \in \mathrm{Hom}(A_a, B)$$

such that $f = \sum_{a=1}^{n} f_a p_a$, and vice versa, f_a uniquely determined f. There are similar decompositions for morphisms in $\mathrm{Hom}(B, A)$. In particular, p_a, q_a themselves are the decomposition of id_A. If B also admits a direct sum decomposition $B = \oplus_{b=1}^{m} B_b$, $u_b \in \mathrm{Hom}(B, B_b)$, $w_b \in \mathrm{Hom}(B_b, B)$, $b = 1, \cdots, m$, satisfying

$$u_a w_b = \delta_{ab}\, \mathrm{id}_{B_a}, \quad \sum_{b=1}^{m} w_b u_b = \mathrm{id}_B, \tag{27}$$

we can even represent $f \in \mathrm{Hom}(A, B)$ by a "matrix":

$$f_{ba} = u_b f q_a \in \mathrm{Hom}(A_a, B_b), \quad f = \sum_{ab} w_b f_{ba} p_a, \tag{28}$$

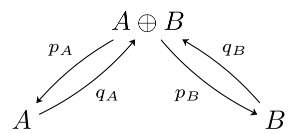
such that composition of morphisms is "matrix" multiplication. The role of decomposition morphisms is like that of basis vectors: $q_a \sim |a\rangle$, $p_a \sim \langle a|$.

The direct sum is automatically compatible with tensor product:

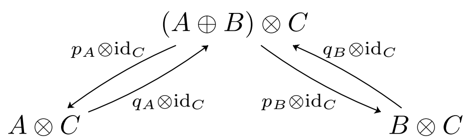
Theorem 6.1 (Distribution law)**.** Let $\mathcal{C}$ be a tensor category, $A, B, C \in \mathcal{C}$, if $A \oplus B$ exists in $\mathcal{C}$,

$$(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$$
$$C \otimes (A \oplus B) \cong (C \otimes A) \oplus (C \otimes B)$$

Proof. Use the following diagram to express the data of a direct sum:



It is easy to check $(A \oplus B) \otimes C$ with the following morphisms is the direct sum of $A \otimes C$ and $B \otimes C$, since we have assumed that $\otimes$ is $\mathbb{C}$-bilinear.



The proof is similar for $C \otimes (A \oplus B) \cong (C \otimes A) \oplus (C \otimes B)$. □

Definition 6.2 (Semisimple)**.** An object A is called simple if $\mathrm{Hom}(A, A) \cong \mathbb{C}$. An object is called semisimple if it is the direct sum of simple objects. A category $\mathcal{C}$ is called semisimple if $\mathcal{C}$ admits finite direct sums and all objects in $\mathcal{C}$ are semisimple.

Let A be a simple object. We denote the canonical isomorphism by

$$\begin{array}{rcl} \mathrm{Hom}(A, A) & \cong & \mathbb{C} \\ f & \mapsto & |f| \quad \text{and } |\,\mathrm{id}_A\,| = 1. \end{array}$$

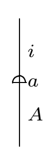
If A is simple, we have $\mathrm{Hom}(A, A) \cong \mathrm{Hom}(\mathbf{1}, A \otimes A^*) \cong \mathrm{Hom}(A^* \otimes A, \mathbf{1}) \cong \mathrm{Hom}(A^*, A^*) \cong \mathbb{C}$, thus A^* is also simple.

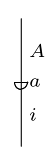
Definition 6.3 (Fusion category)**.** A fusion category is a $\mathbb{C}$-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects and finite dimensional Hom-spaces, and the unit $\mathbf{1}$ is simple.

For a fusion category $\mathcal{C}$ we will change our convention a little, using small letters $i, j, k, \cdots \in L$ (L is the finite set of representing simple objects, one from each isomorphism class) for simple objects. Any object $A \in \mathcal{C}$ is a direct sum of simple objects, i.e. $A = \oplus_{i \in L} i^{\oplus N_i^A}$ and the direct sum decomposition morphisms are denoted by $p_A^{i,a}, q_{i,a}^A, i \in L, a = 1, \cdots, N_i^A$

$$p_A^{i,a} q_{j,b}^A = \delta_{ij} \delta_{ab} \,\mathrm{id}_i, \quad \sum_{i \in L} \sum_{a=1}^{N_i^A} q_{i,a}^A p_A^{i,a} = \mathrm{id}_A \,. \tag{29}$$

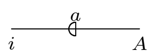
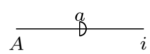
$N_i^A = \dim \mathrm{Hom}(i, A) = \dim \mathrm{Hom}(A, i)$. $N_k^{i \cdots j}$ is short for N_k^A if $A \cong i \otimes \cdots \otimes j$. It is intuitive to use the following graphs for the direct sum decomposition morphisms:

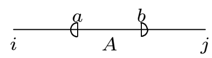


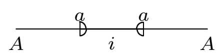
$= p_A^{i,a},$


$= q_{i,a}^A.$ (30)

Our choice of symbol is to remind the reader of the fact that these morphisms serve as the "basis" of A (rotate the graph by 90° anticlockwise). They satisfy similar orthonormal and complete conditions:

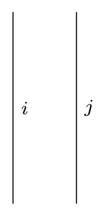
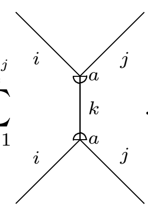
 $\sim \langle i, a|,$  $\sim |a, i\rangle,$ (31)

 $\sim \langle i, a|b, j\rangle = \delta_{ij}\delta_{ab},$ (32)

$$\sum_{ia}$$  $\sim \sum_{ia} |a, i\rangle\langle i, a| = \mathrm{id}_A\,.$ (33)

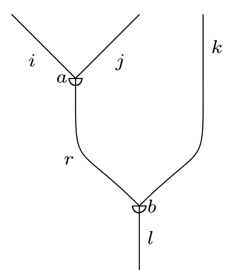
The only subtle part is that the tensor product and braiding involving nontrivial i is different from usual intuitions from vector spaces, which will be explained below.

The data such as tensor product, associator, rigidity of a fusion category $\mathcal{C}$ can be expressed in terms of simple objects. First, the decomposition of the tensor product $i \otimes j$:

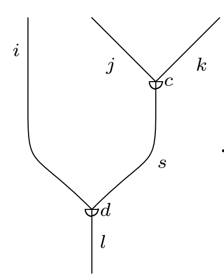
 $= \sum_{k\in L}\sum_{a=1}^{N_k^{ij}}$  . (34)

It is easy to see $N_j^{\mathbf{1}i} = N_j^{i\mathbf{1}} = \delta_{ij}$, and it is convenient to take the corresponding decomposition morphisms as $p_{\mathbf{1}i}^{i,1} = \lambda_i, q_{i,1}^{\mathbf{1}i} = \lambda_i^{-1}, p_{i\mathbf{1}}^{i,1} = \rho_i, q_{i,1}^{i\mathbf{1}} = \rho_i^{-1}$.

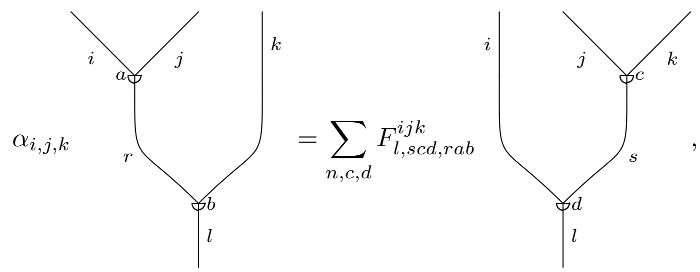
Using Theorem 6.1, the direct sum decomposition of tensor product of more than two objects can be expressed in terms of (34). For example, the embedding morphisms of $(i \otimes j) \otimes k$ are:

 . (35)

Similarly for $i \otimes (j \otimes k)$:

 . (36)

Note that (35)(36) are just bases of $\mathrm{Hom}(l,(i\otimes j)\otimes k)$ and $\mathrm{Hom}(l,i\otimes(j\otimes k))$. The associator isomorphism $\alpha_{i,j,k}$ induces the isomorphism between the two vector spaces, i.e.



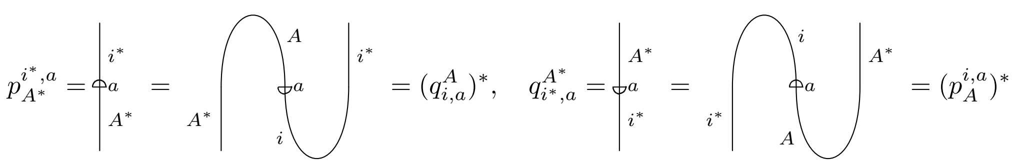
(37)

where F^{ijk}_l is called the F-matrix. Here we write $\alpha_{i,j,k}$ explicitly, but we will omit it below. We also have

$$\sum_m N^{ij}_m N^{mk}_l = \sum_n N^{in}_l N^{jk}_n = \dim \mathrm{Hom}(l,(i\otimes j)\otimes k). \tag{38}$$

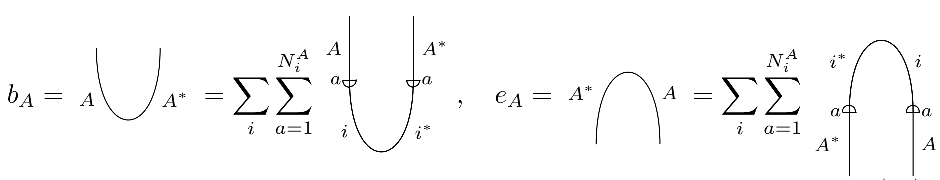
Exercise 6.1. Translate the pentagon equation of the associator α into that of the F-matrix.

Lemma 6.2. For a semisimple rigid tensor category, (A^*, b_A, e_A) is left dual to A. Take the direct sum decomposition morphisms of A^* as



(39)

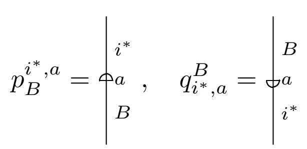
Then, b_A, e_A satisfy



(40)

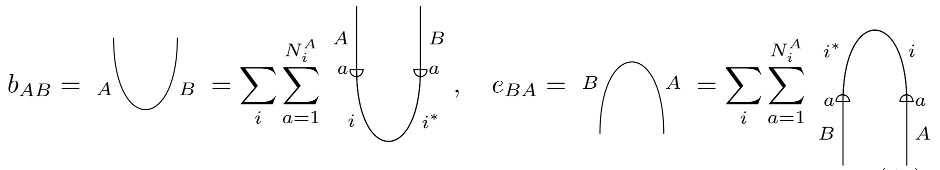
Theorem 6.3. A semisimple tensor category is rigid if all the simple objects have left and right duals.

Proof. Let $A = \oplus_i i^{\oplus N^A_i}$. Take $B = \oplus_i (i^*)^{\oplus N^A_i}$ with decomposition morphisms



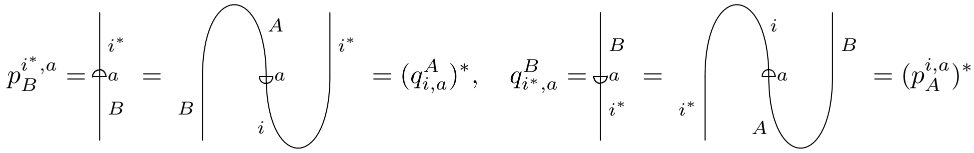
(41)

and define



(42)

Then (B, b_{AB}, e_{BA}) is left dual to A. Moreover, the following equations hold

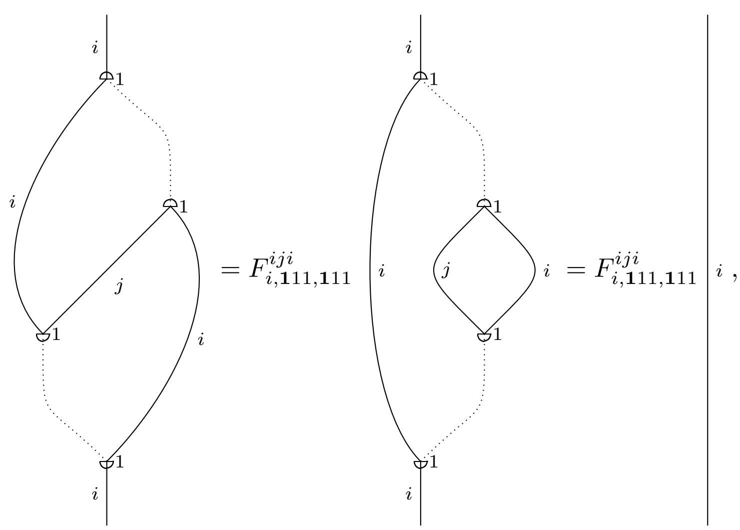


(43)

It is similar to find the right dual to A. □

Theorem 6.4. Let $\mathcal{C}$ be a semisimple category and $\mathbf{1}$ is simple. A simple object $i \in \mathcal{C}$ has left duals if and only if there exists a simple object j, $N_{\mathbf{1}}^{ij} = N_{\mathbf{1}}^{ji} = 1$ and $F^{iji}_{i,\mathbf{1}11,\mathbf{1}11} \neq 0$.

Proof. If there exists a simple object j, $N_{\mathbf{1}}^{ij} = N_{\mathbf{1}}^{ji} = 1$ and $F^{iji}_{i,\mathbf{1}11,\mathbf{1}11} \neq 0$. We can compute the following graph



(44)

as well as a similar graph for $(F^{jij}_j)^{-1}_{\mathbf{1}11,\mathbf{1}11}$. It is an easy exercise to show that $F^{iji}_{i,\mathbf{1}11,\mathbf{1}11} = (F^{jij}_j)^{-1}_{\mathbf{1}11,\mathbf{1}11}$. Then we can take

$$b_i = (F^{iji}_{i,\mathbf{1}11,\mathbf{1}11})^{-1/2} q^{ij}_{\mathbf{1},1}, e_i = (F^{iji}_{i,\mathbf{1}11,\mathbf{1}11})^{-1/2} p^{\mathbf{1},1}_{ji},$$

and (j, b_i, e_i) is left dual to i.

If i has a left dual i^*, recall Corollary 5.2 and we see $\mathrm{Hom}(\mathbf{1}, i \otimes i^*) \cong \mathrm{Hom}(i^* \otimes i, \mathbf{1}) \cong \mathbb{C}$, thus $N_{\mathbf{1}}^{ii^*} = N_{\mathbf{1}}^{i^* i} = 1$ and the differences between b_i, e_i and $q_{\mathbf{1},1}^{ii^*}, p_{i^* i}^{\mathbf{1},1}$ are just nonzero complex numbers. Use similar graphs as eq.(44), we find $F_{i,\mathbf{1}11,\mathbf{1}11}^{ii^* i} = (F_{i^*}^{i^* ii^*})_{\mathbf{1}11,\mathbf{1}11}^{-1} \neq 0$. □

Exercise 6.2. Verify $F_{i,\mathbf{1}11,\mathbf{1}11}^{iji} = (F_j^{jij})_{\mathbf{1}11,\mathbf{1}11}^{-1}$ using the pentagon and triangle equations.

The above established the compatibility between semisimpleness (direct sum) and rigidity. In particular, we can determine if a semisimple category is rigid by the F-matrix. Next, we discuss the unitary structure.

Definition 6.4 (Unitarity)**.** $\mathbb{C}$-linear category $\mathcal{C}$ is called a †-category if for each Hom-space there is an antilinear map

$$\begin{aligned} \dagger : \mathrm{Hom}(A, B) &\rightarrow \mathrm{Hom}(B, A) \\ f &\mapsto f^\dagger \end{aligned}$$

such that

$$\mathrm{id}_A^\dagger = \mathrm{id}_A, \tag{45}$$

$$(gf)^\dagger = f^\dagger g^\dagger, f^{\dagger\dagger} = f. \tag{46}$$

A morphism f in a †-category is called unitary if $f^\dagger = f^{-1}$, and Hermitian if $f^\dagger = f$.

A unitary fusion category (UFC) $\mathcal{C}$ is a fusion †-category satisfying:

1. Positive-definite: for any object A and simple object i, $f \in \mathrm{Hom}(i, A)$ and $f \neq 0$,

$$|f^\dagger f| > 0. \tag{47}$$

2. † is compatible with tensor product:

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger. \tag{48}$$

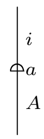
3. The associativity and unit isomorphisms are unitary:

$$\alpha_{A,B,C}^\dagger = \alpha_{A,B,C}^{-1},\ \lambda_A^\dagger = \lambda_A^{-1},\ \rho_A^\dagger = \rho_A^{-1}. \tag{49}$$

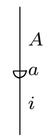
Remark 4. As an intuitive example to explain why we use the symbol †, consider the category **Hilb** whose objects are finite dimensional Hilbert spaces and morphisms are linear operators (or usual complex matrices if you prefer to work with a chosen basis). **Hilb** is a unitary fusion category, whose † is just the usual Hermitian conjugate.

The following establishes the compatibility between unitarity and semisimpleness.

Theorem 6.5. For a unitary fusion category, one can always choose the decomposition of object A as



$$= p_A^{i,a} = (q_{i,a}^A)^\dagger = \left(\quad \right.$$



$$\left. \quad \right)^\dagger, \tag{50}$$

which we call an orthonormal decomposition.

Proof. Suppose that we have a decomposition $p_A^{i,a}, q_{i,a}^A$ (may not satify the condition in the theorem). Note that $q_{i,a}^A$ form a basis of $\mathrm{Hom}(i, A)$, and $(p_A^{i,a})^\dagger$ can be expressed in terms of $q_{i,a}^A$,

$$(p_A^{i,a})^\dagger = \sum_b P_{ab} q_{i,b}^A. \tag{51}$$

This is, in general, how the unitarity data is expressed in terms of simple objects. It is not as important as F, R-matrices since it can be made trivial simply by change of basis.

Using the properties of decomposition morphisms (29),

$$P_{ab}\,\mathrm{id}_i = p_A^{i,b}(p_A^{i,a})^\dagger. \tag{52}$$

Thus

$$\overline{P_{ab}}\,\mathrm{id}_i = (P_{ab}\,\mathrm{id}_i)^\dagger = (p_A^{i,b}(p_A^{i,a})^\dagger)^\dagger = p_A^{i,a}(p_A^{i,b})^\dagger = P_{ba}\,\mathrm{id}_i, \tag{53}$$

namely P is a Hermitian matrix, $P^\dagger = P$. Suppose that P is diagonized by unitary matrix U, $(U^\dagger P U)_{ab} = \lambda_a \delta_{ab}$,

$$\lambda_a\,\mathrm{id}_i = \sum_b \overline{U_{ba}} p_A^{i,c}(p_A^{i,b})^\dagger U_{ca} = \left(\sum_b U_{ba} p_A^{i,b}\right)\left(\sum_b U_{ba} p_A^{i,b}\right)^\dagger, \tag{54}$$

by (47), we know that $\lambda_a > 0$, namely P is positive definite.

We seek for different bases:

$$u_A^{i,a} = \sum_b X_{ab} p_A^{i,b}, \quad w_{i,a}^A = \sum_b W_{ab} q_{i,b}^A. \tag{55}$$

Such that they are still decomposition mophisms

$$\delta_{ab}\,\mathrm{id}_i = u_A^{i,a} w_{i,b}^A = \sum_c X_{ac} W_{bc}\,\mathrm{id}_i, \tag{56}$$

$$\mathrm{id}_A = \sum_i \sum_a w_{i,b}^A u_A^{i,a} = \sum_i \sum_{bc} \sum_a W_{ab} X_{ac} q_{i,b}^A p_A^{i,c}. \tag{57}$$

Therefore, we need

$$W^{-1} = X^T. \tag{58}$$

Thus $q_{i,b} = \sum_d X_{db} w^A_{i,d}$, and we have

$$(u^{i,a}_A)^\dagger = \sum_{bcd} \overline{X_{ac}} P_{cb} X_{db} w^A_{i,d} = \sum_b (\overline{X} P X^T)_{ab} w^A_{i,b}. \tag{59}$$

Therefore, taking $X_{ab} = \lambda_a^{-1/2} U_{ba}$ will make the new bases $u^{i,a}_A, w^A_{i,a}$ satisfy the conditions in the theorem. Such choice of decomposition morphisms will always be assumed below. □

After fixing decompositions $p^{i,a}_A$, we can represent a morphism $f \in \mathrm{Hom}(A, B)$ by a block-diagonal matrix

$$f_{i,ba} = |p^{i,b}_B f (p^{i,a}_A)^\dagger|, \quad f = \sum_{iab} f_{i,ba} (p^{i,b}_B)^\dagger p^{i,a}_A, \tag{60}$$

where the simple object i labels the block. Then composition of morphisms is simply matrix multiplication, and † is taking Hermitian conjugate. In particular, we have

Corollary 6.6. In a unitary fusion category, if a morphism $f \in \mathrm{Hom}(A, B)$ satisfies $f^\dagger f = 0$, we must have $f = 0$.

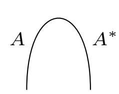
Remark 5. The positive-definite condition in the definition of UFC can be replaced by the †-definite condition: $f^\dagger f = 0 \Rightarrow f = 0$ together with requirement that for any $f \in \mathrm{Hom}(A, B)$, there exists $a \in \mathrm{Hom}(A, A)$ such that $f^\dagger f = a^\dagger a$.

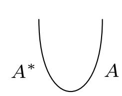
Corollary 6.7. F^{ijk}_l is just the l-labeled block matrix of morphism $\alpha_{i,j,k} : (i \otimes j) \otimes k \to i \otimes (j \otimes k)$,

$$F^{ijk}_{l,scd,rab} = (\alpha_{i,j,k})_{l,scdrab} = \left| p^{l,d}_{is} (\mathrm{id}_i \otimes p^{s,c}_{jk}) \alpha_{i,j,k} (p^{r,a}_{ij} \otimes \mathrm{id}_k)^\dagger (p^{l,b}_{rk})^\dagger \right|, \tag{61}$$

thus it is a unitary matrix.

Next, we discuss the interplay between unitarity and rigidity. We use the following graphs for $b^\dagger_A, e^\dagger_A$,

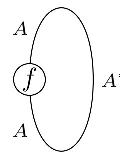
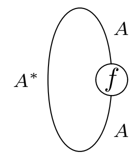


$= b^\dagger_A$,


$= e^\dagger_A$, (62)

and one can check that $(A^*, e^\dagger_A, b^\dagger_A)$ is right dual to A by taking † map of the graphs (24). Also $(A, e^\dagger_A, b^\dagger_A)$ is left dual to A^*, thus $A^{**} \cong A$, and we are able to define the *trace* of morphisms:

Definition 6.5 (Quantum trace)**.** The right and left quantum trace of $f \in \mathrm{Hom}(A, A)$ is

$$\mathrm{tr}_R\, f = |b_A^\dagger (f \otimes \mathrm{id}_{A^*}) b_A| =$$  $$, \quad \mathrm{tr}_L\, f = |e_A (\mathrm{id}_{A^*} \otimes f) e_A^\dagger| =$$  $$, \tag{63}$$

where we omitted || for unit object $\mathbf{1}$ (for a closed graph).

Definition 6.6 (Spherical)**.** A fusion category is called spherical if $\mathrm{tr}_L\, f = \mathrm{tr}_R\, f$ for any morphism f.

Note that although b_A, e_A must compose to id_A or id_{A^*}, there is a degree of freedom left: one can always scale them by the same complex number $b'_A = \lambda b_A, e'_A = \lambda^{-1} e_A$. To fix such ambiguity, we would like to choose the following normalization convention:

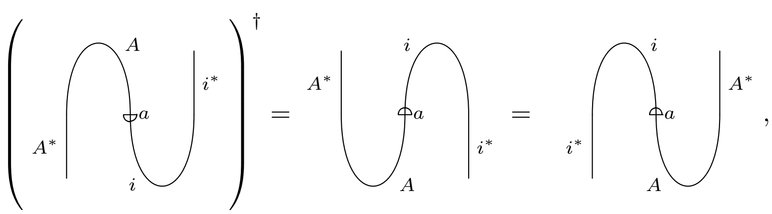
$$\mathrm{tr}_R\, \mathrm{id}_A = |b_A^\dagger b_A| = |e_A e_A^\dagger| = \mathrm{tr}_L\, \mathrm{id}_A = d_A. \tag{64}$$

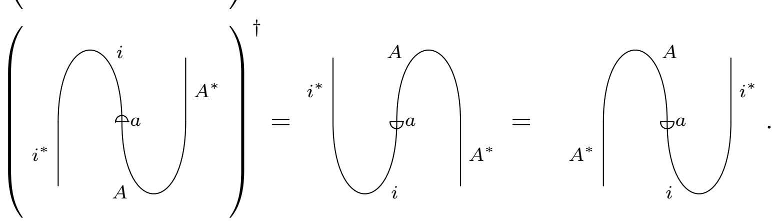
The positive number d_A in (64) is called the quantum dimension of object A. This choice makes the unitary fusion category *spherical*, as explained in the next theorem. Since $\mathrm{tr}_R\, \mathrm{id}_A = \mathrm{tr}_L\, \mathrm{id}_{A^*}$, we see $d_A = d_{A^*}$. Recall (44) we know that $d_i = |F^{ii^*i}_{i,\mathbf{1}\mathbf{1}\mathbf{1},\mathbf{1}\mathbf{1}\mathbf{1}}|^{-1}$.

Theorem 6.8. For an object A in a UFC $\mathcal{C}$ the following three are equivalent

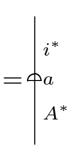
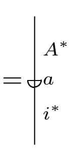
(a) $\forall f \in \mathrm{Hom}(A, A), \mathrm{tr}_R\, f = \mathrm{tr}_L\, f$.

(b) For the direct sum decomposition morphisms $p_A^{i,a}$ of A

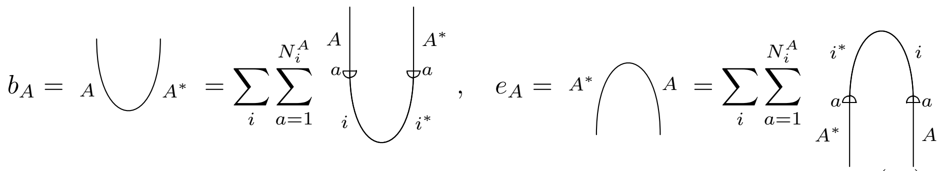
 $$, \tag{65}$$

 $$. \tag{66}$$

(c) There exist decomposition morphisms of A^*

$$p_{A^*}^{i^*,a} =$$  $$, \quad (p_{A^*}^{i^*,a})^\dagger =$$  $$, \tag{67}$$

such that

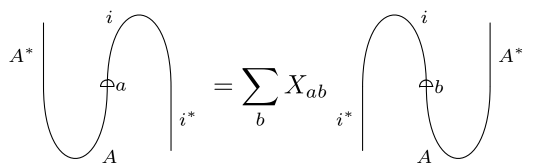


(68)

Proof. (b)⇔(c) Similar as in Lemma 6.2 and Theorem 6.3.

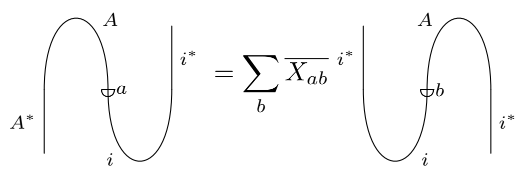
(c)⇒(a) By straightforward calculation $\mathrm{tr}_L f = \mathrm{tr}_R f = \sum_i \sum_{a=1}^{N_i^A} |p_A^{i,a} f (p_A^{i,a})^\dagger| d_i$. This is a direct consequence of making the choice (64) for all simple objects i.

(a)⇒(b) Say



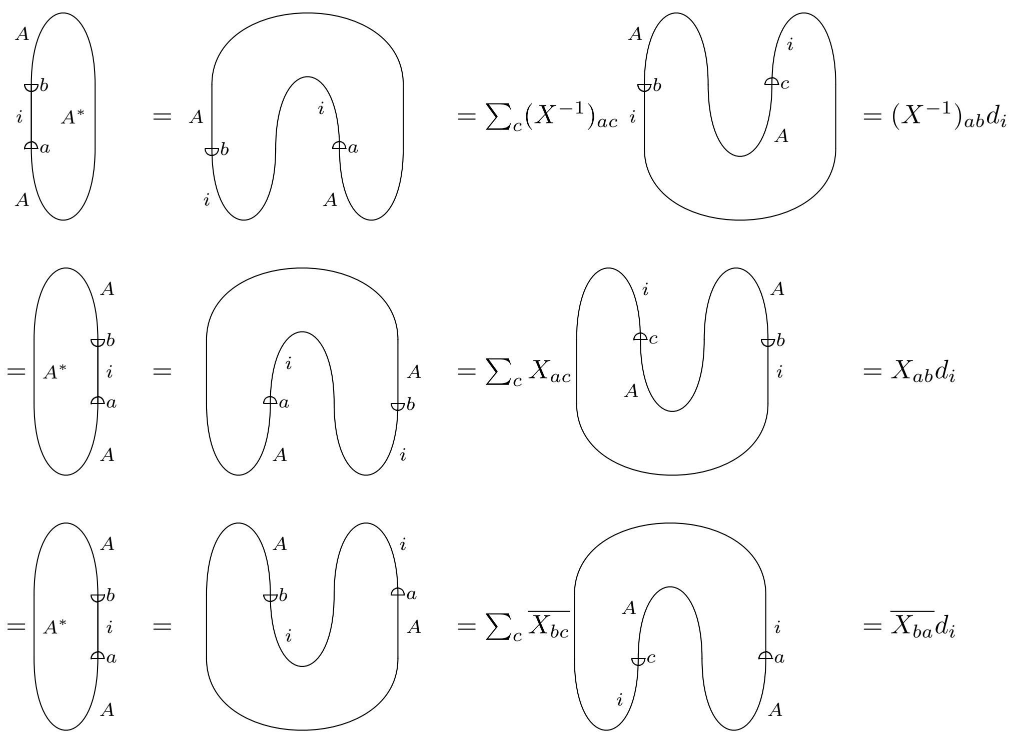
(69)

X is an invertible matrix. Take † map



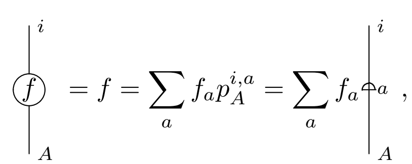
(70)

Then consider the following graphs



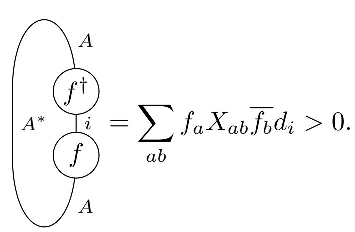
(71)

We conclude $X^{-1} = X = X^{\dagger}$. Moreover, for any non-zero morphism $f \in \mathrm{Hom}(A, i)$,



(72)

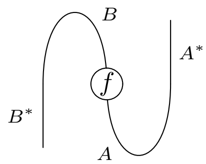
we have



(73)

Thus X is positive definite, together with $X^{-1} = X = X^{\dagger}$ we know that X is identity matrix, $X_{ab} = \delta_{ab}$. □

Corollary 6.9. If A, B both satisfy the conditions in Theorem 6.8, $\forall f \in \mathrm{Hom}(A, B)$



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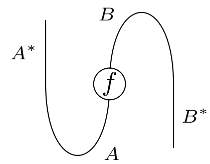
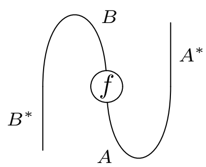
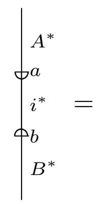


. (74)

Proof. By straightforward calculation



$$= \sum_i \sum_{a=1}^{N_i^A} \sum_{b=1}^{N_i^B} |(p_B^{i,b})^\dagger f p_A^{i,a}|$$


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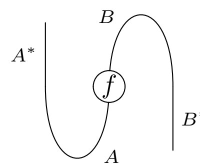
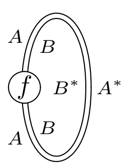


. (75)

□

Corollary 6.10. If A, B both satisfy the conditions in Theorem 6.8, $A \otimes B$ also satisfy those conditions.

Proof.



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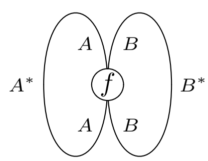

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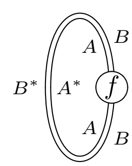
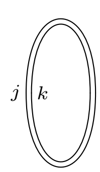


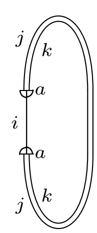
. (76)

□

Therefore, as long as we choose structure morphisms like in Theorem 6.8(c), the unitary fusion category is made spherical. In the followings we will drop the subscript of quantum trace and write $\mathrm{tr} = \mathrm{tr}_L = \mathrm{tr}_R$. The Hom spaces of a unitary fusion category are Hilbert spaces whose inner product is given by the trace. Assuming $f, g \in \mathrm{Hom}(A, B)$, the inner product is given by $\langle f|g\rangle \propto \mathrm{tr}\, f^\dagger g$, up to a normalization factor one can choose for convenience.

The formula (71) in the proof of Theorem 6.8 is quite useful. Taking $A = j \otimes k$ and sum over i and a, b, we obtain a key equation for quantum dimensions d_i.

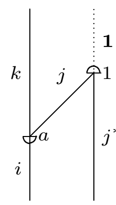
$$d_j d_k = \mathrm{tr}\,\mathrm{id}_{j\otimes k} =$$


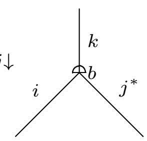
$$= \sum_i \sum_{a=1}^{N_i^{jk}}$$


$$= \sum_i N_i^{jk} d_i. \quad (77)$$

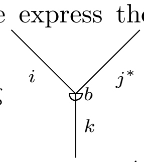
Since N_i^{jk} are non-negative integers, by the Perron–Frobenius theorem, one knows that d_i is the largest positive eigenvalue of the matrix N^i whose entries are $(N^i)_{jk} = N_k^{ij}$.

Another application is the "rotatable vertex". The "orthonormal" basis vertices we have been using has a disadvantage: if one tries to rotate them, by adding caps or cups to bend the legs, the resulting vertex is no longer normalized. This can be seen by analyzing the following graph:



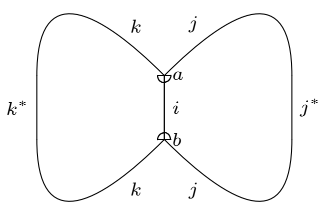
$$= \sum_b X_{ab}^{i;kj\downarrow}$$


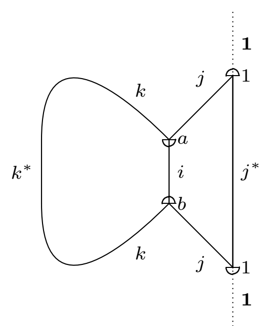
, (78)

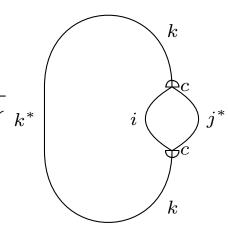
where we express the left side in terms of the basis on the right side. By attaching 

to the bottom, it is easily found that $X_{ab}^{i;kj\downarrow} = F^{kjj^*}_{k,\mathbf{1}11,iab}$.

Next we compute $X_{ab}^{i;kj\downarrow}$ via the following graph,

$$d_i \delta_{ab} =$$


$$= d_j$$


$$= d_j \sum_c X_{ac}^{i;kj\downarrow} \overline{X_{bc}^{i;kj\downarrow}}$$


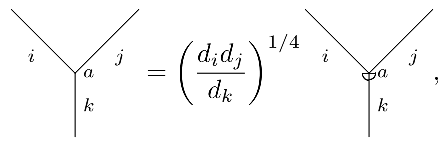
$$= d_j d_k \sum_c X_{ac}^{i;kj\downarrow} \overline{X_{bc}^{i;kj\downarrow}}, \quad (79)$$

where we have used (71) by setting $A = k \otimes j$. We conclude that

$$X_{ab}^{i;kj\downarrow} = F^{kjj^*}_{k,\mathbf{1}11,iab} = \sqrt{\frac{d_i}{d_j d_k}} U_{ab}^{i;kj\downarrow}, \quad (80)$$

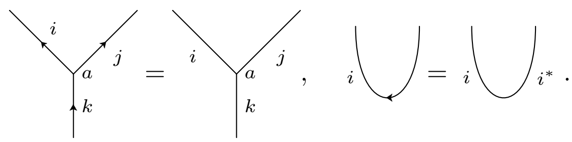
where $U_{ab}^{i;kj\downarrow}$ is a unitary matrix.

Therefore, if we rescale our basis



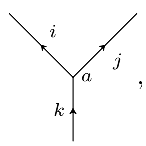
(81)

the bending like in (78) will leads to only a unitary matrix. At the same time, we like to change the graphs to be directed, to remind ourselves whether a vertex has been rotated or not.

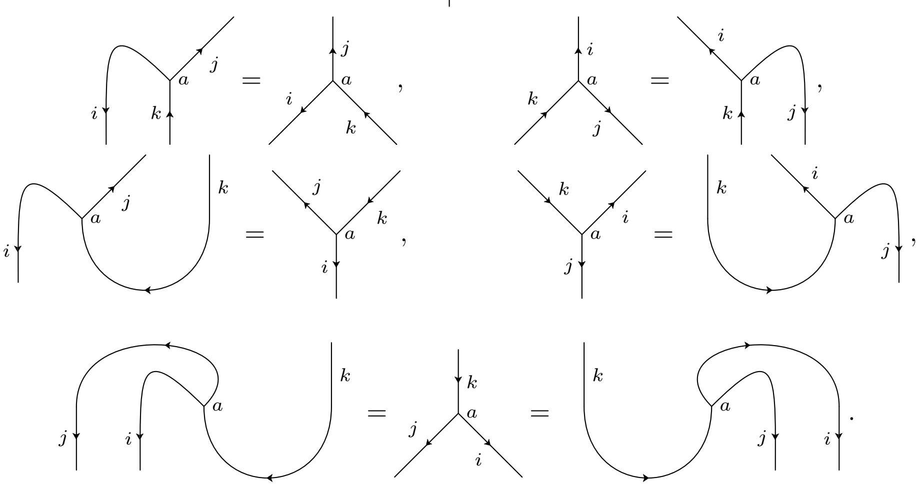


(82)

To avoid ambiguity, we require that all the vertices are *branched*: we only allow vertices with two incoming legs and one outgoing leg or with one incoming leg and two outgoing legs; vertices with three incoming legs or three outgoing legs are forbidden. A rotated vertex is related to the original one (put all arrows going upwards) in the following way:

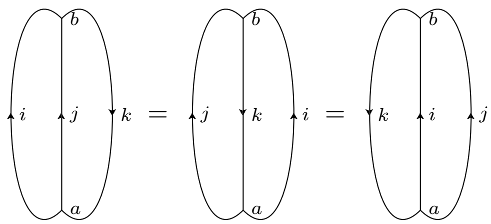


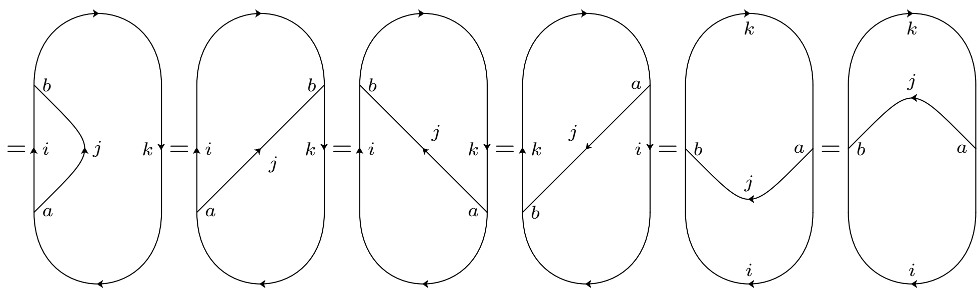
(83)



For a single vertex, rotating leads to a unitary matrix. In most cases, such a unitary matrix can be set to identity by properly choosing the basis. However, sometimes we do not have enough degrees of freedom to do so. For example, if $N_{i^*}^{ii} = 1$, we can only choose one vector in $\mathrm{Hom}(i \otimes i, i^*)$ as our basis, and rotating it as (78) must give a phase factor that is not removable by change of basis.

Such a phase factor is thus an invariance, which is known as the $\mathbb{Z}_3$ Frobenius-Shur indicator. In this case, our branched directed convention actually gives a overcomplete basis, since a rotated vertex is just the original one times a phase factor, but we use different graphs to represent them. However, if we are considering a closed graph and rotating the vertices in it all together, the unitary matrices automatically cancels each other. Therefore the branched directed convention is very convenient for calculating closed graphs. For example,

$\delta_{ab}\sqrt{d_i d_j d_k} =$ 



$= \cdots$ (all topologically equivalent graphs).

(84)

In the following we may use this branched directed convention which allows rotation of the graphs, or the original bottom-top undirected convention which has to be read from bottom to top. It is natural to interpret the wave function of string-net model by graphical calculus of unitary fusion category.

Example 6.1. We now discuss an important example of unitary fusion category, the category of representations of a finite group G, denoted by $\operatorname{Rep} G$. (Here the finite condition of the group is due to the finite requirement of fusion category. The construction below applies to general group but the resulting category may not be finite.)

The compact definition is that $\operatorname{Rep} G := \operatorname{Fun}(BG, \mathbf{Hilb})$, where BG is the category with only one object $\star$ and $\operatorname{Hom}_{BG}(\star,\star) = G$, whose composition is defined by the multiplication of G. Now we unpack the this definition and elaborate on the fusion structures.

The objects of $\operatorname{Rep} G$ are pairs (V, ρ), where V is a Hilbert space (the image of $\star$) and ρ is the map

$$\begin{aligned} \rho : \operatorname{Hom}(\star,\star) = G &\to GL(V) \subset \operatorname{Hom}(V,V), \\ g &\mapsto \rho_g \in \operatorname{Hom}(V,V), \end{aligned}$$

satisfying $\rho_{gh} = \rho_g \rho_h$. In other words, ρ is a group homomorphism.

A morphism from (V, ρ) to (W, τ) is a linear map $f : V \to W$ satisfying the following condition (the natural square)

$$\forall g \in G, \quad \begin{array}{ccc} V & \xrightarrow{f} & W \\ \big\downarrow \rho_g & & \big\downarrow \tau_g \\ V & \xrightarrow{f} & W \end{array}$$

$$\tau_g f = f \rho_g.$$

Morphisms in $\operatorname{Rep} G$ are also called intertwiners, (G-)invariant tensors, or (G-)symmetric tensors. Indeed, the graphical calculus in $\operatorname{Rep} G$ are nothing but a G-symmetric tensor networks.

The direct sum and semisimple structure follows from the the property that every finite group representation is completely reducible. The tensor product of representations is defined as follows:

$$(V, \rho) \otimes (W, \tau) := (V \otimes W, \rho \otimes \tau),$$
$$(\rho \otimes \tau)_g = \rho_g \otimes \tau_g.$$

The tensor unit is the trivial representation $(\mathbb{C}, 1)$, $1_g = 1, \forall g$. The dual representation of (V, ρ) is $(V^* = \operatorname{Hom}(V, \mathbb{C}), \rho^*)$, $(\rho^*)_g = (\rho_{g^{-1}})^*$. The unitary structure is given by the usual Hermitian conjugate.

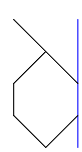
A well known physical exercise that involves both the direct sum and tensor structure is the addition of angular momentum, viewed as representations of $SU(2)$ (although it is not finite group). One has

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|.$$

The p, q maps of the direct sum structure are encoded in the Clebsch-Gordon coefficients or the Wigner 3j-symbols, and F^{ijk}_l matrices are just the Wigner 6j-symbols up to proper normalization.

7 Module Category

We make cut the string-net wavefunction in half and get a boundary like



This kind of boundary wave function is described the theory of module category.

Definition 7.1 (Module category)**.** A left module category $\mathcal{M}$ over a tensor category $\mathcal{C}$ is given by

1. A functor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$,
2. Functorial associativity and unit isomorphisms: $\alpha_{A,B,M} : (A \otimes B) \otimes M \to A \otimes (B \otimes M), \lambda_M : \mathbf{1} \otimes M \to M$ for any $A, B \in \mathcal{C}, M \in \mathcal{M}$,

Such that $\mathrm{Pen}_{A,B,C,M}$ and $\mathrm{Tri}_{A,\mathbf{1},M}$ commute, i.e.,

1. Pentagon equations: $\forall A, B, C \in \mathcal{C}, M \in \mathcal{M}$ the diagram

$$\begin{array}{ccc} ((A \otimes B) \otimes C) \otimes M & \xrightarrow{\alpha_{A,B,C} \otimes \mathrm{id}_M} & (A \otimes (B \otimes C)) \otimes M \\ \downarrow{\scriptstyle \alpha_{A \otimes B,C,M}} & & \downarrow{\scriptstyle \alpha_{A,B \otimes C,M}} \\ (A \otimes B) \otimes (C \otimes M) & & A \otimes ((B \otimes C) \otimes M) \\ & {\scriptstyle \alpha_{A,B,C \otimes M}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \alpha_{B,C,M}} & \\ & A \otimes (B \otimes (C \otimes M)) & \end{array} \tag{85}$$

commutes

2. Triangle equations: $\forall A \in \mathcal{C}, M \in \mathcal{M}$ the diagram

$$\begin{array}{ccc} (A \otimes \mathbf{1}) \otimes M & \xrightarrow{\alpha_{A,\mathbf{1},M}} & A \otimes (\mathbf{1} \otimes M) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_M} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_M} & \\ & A \otimes M & \end{array} \tag{86}$$

commutes.

Definition 7.2 (Module functor)**.** Let $\mathcal{M}$ and $\mathcal{N}$ be two left module categories over a tensor category $\mathcal{C}$. A left module functor from $\mathcal{M}$ to $\mathcal{N}$ is a pair (F, β), a functor $F : \mathcal{M} \to \mathcal{N}$ with natural isomorphisms $\beta_{A,M} : F(A \otimes M) \to A \otimes F(M)$, satisfying

1. Pentagon equations: $\forall A, B \in \mathcal{C}, M \in \mathcal{M}$ the diagram

$$\begin{array}{ccc} F((A \otimes B) \otimes M) & \xrightarrow{F(\alpha_{A,B,M})} & F(A \otimes (B \otimes M)) \\ \downarrow{\scriptstyle \beta_{A \otimes B,M}} & & \downarrow{\scriptstyle \beta_{A,B \otimes M}} \\ (A \otimes B) \otimes F(M) & & A \otimes F(B \otimes M) \\ & {\scriptstyle \alpha_{A,B,F(M)}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \beta_{B,M}} & \\ & A \otimes (B \otimes F(M)) & \end{array} \tag{87}$$

commutes

2. Triangle equations: $\forall M \in \mathcal{M}$ the diagram

$$\begin{array}{ccc} F(\mathbf{1}\otimes M) & \xrightarrow{\beta_{\mathbf{1},M}} & \mathbf{1}\otimes F(M) \\ & {\scriptstyle F(\lambda_M)}\searrow \quad \swarrow{\scriptstyle \lambda_{F(M)}} & \\ & F(M) & \end{array} \tag{88}$$

commutes.

Right module categories and functors are similarly defined (when left/right omitted I mean left). In particular, right $\mathcal{C}$-module is the same as left $\mathcal{C}^{\text{rev}}$-module. A natural transformation ν between two module functor (F,β) and (F',β') should satisfy the additional condition that the diagram

$$\begin{array}{ccc} F(A\otimes M) & \xrightarrow{\nu_{A\otimes M}} & F'(A\otimes M) \\ {\scriptstyle \beta_{A,M}}\downarrow & & \downarrow{\scriptstyle \beta'_{A,M}} \\ A\otimes F(M) & \xrightarrow[\mathrm{id}_A\otimes\nu_M]{} & A\otimes F'(M) \end{array} \tag{89}$$

commutes. Module functors between two module categories $\mathcal{M},\mathcal{N}$ over a tensor category $\mathcal{C}$ also form a category, denoted by $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{N})$. We have the following theorem:

Theorem 7.1. Let $\mathcal{C}$ be a tensor category and $\mathcal{M}$ a module category over $\mathcal{C}$. The category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{M})$ is equivalent to $\mathcal{M}$.

Proof. We have such two functors

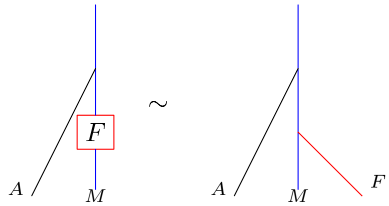
$$\begin{aligned} &\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{M})\to\mathcal{M}, && F\mapsto F(\mathbf{1}), && \nu\mapsto\nu_{\mathbf{1}}, \\ &\mathcal{M}\to\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{M}), && M\mapsto -\otimes M, && f\mapsto \mathrm{id}_{-}\otimes f. \end{aligned}$$

It is easy to check that they give the equivalence $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{M})\simeq\mathcal{M}$. ☐

Note that $\mathcal{C}$ is a module category over itself. $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})$ is moreover a tensor category (tensor product is the composition of functors, $(F\otimes G)(-)=F(G(-))$), and the above is also an equivalence of tensor categories $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})\simeq\mathcal{C}^{\text{rev}}$. $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{M})$ is a right module over $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C})$, which completely copies the structure of $\mathcal{M}$ being a left module over $\mathcal{C}$. Similarly, $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$ is also a tensor category, and $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{M})$ has a natural left module structure over $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$, namely, $\mathcal{M}$ has a natural right module structure over $(\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M}))^{\text{rev}}$, given by

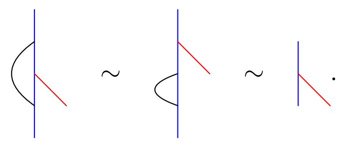
$$M\otimes F := F(M),\quad (A\otimes M)\otimes F \xrightarrow{\beta_{A,M}} A\otimes(M\otimes F). \tag{90}$$

This fact allows us to present module functors graphically like



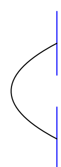
(91)

Physically, $\mathcal{C}$ describes the wavefunction of the left bulk, left modules $\mathcal{M}, \mathcal{N}, \cdots$ over $\mathcal{C}$ describes the wavefunctions of gapped boundaries. (91) shows two ways to interpret the graph of module functors. On one hand, $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ can be viewed as a 0+1D defect on the boundary between $\mathcal{M}, \mathcal{N}$, in particular, $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ describes the quasiparticles on the boundary $\mathcal{M}$. The following wavefunction renormalization, based on the data of $\beta_{A,M}$, can be performed



(92)

Intuitively, the defect/quasiparticle on the boundary absorbs a half loop of wavefunction in the left bulk. Indeed, the graph



(93)

carries a *weak Hopf algebra* structure[12], and module functors form modules over it. This observation allows a finite algorithm computing the module functors, i.e. the defects/quasiparticles, from the ground state wavefunction.

On the other hand, one can also interpret the graph of module functors on the right as wavefunction of another bulk phase described by $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ on the right. This interpretation is more "symmetric"; we call the tensor category $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ dual to $\mathcal{C}$ with respect to $\mathcal{M}$, denoted by $\mathcal{C}_{\mathcal{M}}^{\vee}$, for which the reason will become clear soon. First, $\mathcal{C}$ has a natural structure of left module functor on $\mathcal{M}$ viewed as a left module over $\mathcal{C}_{\mathcal{M}}^{\vee}$, by left action

$$\mathcal{C} \to \mathrm{Fun}_{\mathcal{C}_{\mathcal{M}}^{\vee}}(\mathcal{M}, \mathcal{M}), \quad A \mapsto A \otimes -. \tag{94}$$

And $\mathrm{Fun}_{\mathcal{C}_{\mathcal{M}}^{\vee}}(\mathcal{M}, \mathcal{M})$ is just the category dual to $\mathcal{C}_{\mathcal{M}}^{\vee}$ with respect to $\mathcal{M}$. Indeed, the above functor is an equivalence (to prove this in general we may need to make some reasonable technical assumptions. For the examples we are interested in this chapter, where $\mathcal{C}$ is a unitary fusion category and $\mathcal{M}$ is semisimple and indecomposable, it is always true), taking dual twice gives us the original category:

$$\mathcal{C} \simeq (\mathcal{C}_{\mathcal{M}}^{\vee})_{\mathcal{M}}^{\vee} = \mathrm{Fun}_{\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})}(\mathcal{M}, \mathcal{M}). \tag{95}$$

Note that module functors, $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M})$, are roughly speaking the functors that "communte" with the action of $\mathcal{C}$, thus taking module functors is like taking the commutants of $\mathcal{C}$ in $\mathrm{Fun}(\mathcal{M},\mathcal{M})$. The above equation is categorically "taking double commutants".

An easy observation is that $\mathcal{M} \simeq \mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{M})$ is simultaneously a right module over $\mathrm{Fun}_{\mathcal{C}}(\mathcal{C},\mathcal{C}) \simeq \mathcal{C}^{\mathrm{rev}}$ and a left module over $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M},\mathcal{M}) = \mathcal{C}^{\vee}_{\mathcal{M}}$; in fact, it is a bimodule

Definition 7.3 (Bimodule category)**.** Let $\mathcal{C},\mathcal{C}'$ be two tensor categories. A $\mathcal{C}$-$\mathcal{C}'$-bimodule category $\mathcal{M}$ is simultaneously a left $\mathcal{C}$-module category and a right $\mathcal{C}'$-module category, with additional functorial associativity isomorphisms

$$\alpha_{A,M,A'} : (A \otimes M) \otimes A' \to A \otimes (M \otimes A')$$

for all $A \in \mathcal{C}, M \in \mathcal{M}, A' \in \mathcal{C}'$, such that the $\mathrm{Pen}_{A,B,M,A'}$ and $\mathrm{Pen}_{A,M,A',B'}$ diagrams commute for all $A, B \in \mathcal{C}, A', B' \in \mathcal{C}'$.

Definition 7.4 (Bimodule functor)**.** Let $\mathcal{C}$,$\mathcal{D}$ be two tensor categories, and $\mathcal{M}$,$\mathcal{N}$ be two $\mathcal{C}$-$\mathcal{D}$-bimodule categories. A bimodule functor from $\mathcal{M}$ to $\mathcal{N}$ is a triple (F,β,γ) such that (F,β), (F,γ) are left, right module functors respectively and the diagram

$$\begin{array}{ccc} F((A\otimes M)\otimes B) & \xrightarrow{F(\alpha^{\mathcal{M}}_{A,M,B})} & F(A\otimes(M\otimes B)) \\ \downarrow{\scriptstyle \gamma_{A\otimes M,B}} & & \downarrow{\scriptstyle \beta_{A,M\otimes B}} \\ F(A\otimes M)\otimes B & & A\otimes F(M\otimes B) \\ \downarrow{\scriptstyle \beta_{A,M}\otimes \mathrm{id}_B} & & \downarrow{\scriptstyle \mathrm{id}_A\otimes\gamma_{M,B}} \\ (A\otimes F(M))\otimes B & \xrightarrow[\alpha^{\mathcal{N}}_{A,F(M),B}]{} & A\otimes(F(M)\otimes B) \end{array} \tag{96}$$

commutes for any $A \in \mathcal{C}, M \in \mathcal{M}, B \in \mathcal{D}$.

Similarly all the bimodule functors from $\mathcal{M}$ to $\mathcal{N}$ form a category, denoted by $\mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M},\mathcal{N})$. Distinguished from generic bimodules, $\mathcal{M}$ as $\mathcal{C}^{\vee}_{\mathcal{M}}$-$\mathcal{C}^{\mathrm{rev}}$-bimodule is *invertible*. The following definitions explain this notion:

Definition 7.5 (Balanced functor)**.** Let $\mathcal{M}$ be a right $\mathcal{C}$-module category, $\mathcal{N}$ a left $\mathcal{C}$-module category and $\mathcal{O}$ a linear category, a $\mathcal{C}$-balanced functor from $\mathcal{M}\times\mathcal{N}$ to $\mathcal{O}$ is a pair $(\Box,\alpha)$, where $\Box$ is a bilinear functor

$$\begin{aligned} \Box : \mathcal{M}\times\mathcal{N} &\to \mathcal{O} \\ (M,N) &\mapsto M\Box N, \end{aligned}$$

and α is a natural isomorphism between functors $\Box(\otimes\times\mathrm{id}_{\mathcal{N}}) \to \Box(\mathrm{id}_{\mathcal{M}}\times\otimes)$, $\alpha_{M,C,N} : (M\otimes C)\Box N \to M\Box(C\otimes N)$ such that $\mathrm{Pen}_{M,C,C',N}$, $\mathrm{Tri}_{M,1,N}$ diagrams commute for all $M\in\mathcal{M}, N\in\mathcal{N}, C, C'\in\mathcal{C}$.

Definition 7.6 (Tensor product of module categories)**.** Let $\mathcal{M}$ be a right $\mathcal{C}$-module category and $\mathcal{N}$ a left $\mathcal{C}$-module category, the tensor product of $\mathcal{M}$ and $\mathcal{N}$ over $\mathcal{C}$ is a linear category $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ with a $\mathcal{C}$-balanced functor $\boxtimes_{\mathcal{C}} : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ such that any $\mathcal{C}$ balanced functor $\Box : \mathcal{M} \times \mathcal{N} \to \mathcal{O}$ uniquely factors through $\boxtimes_{\mathcal{C}}$, i.e.

$$\begin{array}{ccc} \mathcal{M} \times \mathcal{N} & & \\ {\scriptstyle \boxtimes_{\mathcal{C}}}\downarrow & \searrow^{\Box} & \\ \mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N} & \xrightarrow[F]{} & \mathcal{O} \end{array} \tag{97}$$

where F is the unique linear functor satisfying $F \boxtimes_{\mathcal{C}} = \Box$.

Remark 6. If moreover, $\mathcal{M}$ is a $\mathcal{D}$-$\mathcal{C}$-bimodule category and $\mathcal{N}$ is a $\mathcal{C}$-$\mathcal{E}$-bimodule category, their tensor product $\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$ is automatically a $\mathcal{D}$-$\mathcal{E}$-bimodule category.

Remark 7. Note that fusion categories can all be viewed as module categories over **Vec**. In this case $\boxtimes_{\mathbf{Vec}}$ recovers the usual Deligne tensor product, and the subscript will be omitted. The notion of $\mathcal{C}$-$\mathcal{D}$-bimodule is the same as that of $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{rev}}$-module.

Definition 7.7 (Invertible bimodule and Morita equivalence)**.** Let $\mathcal{C}, \mathcal{D}$ be tensor categories, and $\mathcal{M}$ be $\mathcal{C}$-$\mathcal{D}$-bimodule. $\mathcal{M}$ is called invertible if $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{M}^{\mathrm{op}} \simeq \mathcal{C}, \mathcal{M}^{\mathrm{op}} \boxtimes_{\mathcal{C}} \mathcal{M} \simeq \mathcal{D}$ as bimodules, and meanwhile $\mathcal{C}$ and $\mathcal{D}$ are called Morita equivalent.

Remark 8. We have a tensor functor $X \mapsto - \otimes X : \mathcal{D}^{\mathrm{rev}} \to \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M}) = \mathcal{C}^{\vee}_{\mathcal{M}}$. $\mathcal{M}$ is invertible if and only if such functor is an equivalence.[8] Therefore, for invertible bimodule $\mathcal{M}$ it suffices to consider $\mathcal{M}$ as $\mathcal{C} \boxtimes \mathcal{C}^{\vee}_{\mathcal{M}}$-module.

Another more physical way to understand the special properties of $\mathcal{M}$ as an invertible $\mathcal{C}^{\vee}_{\mathcal{M}}$-$\mathcal{C}^{\mathrm{rev}}$-bimodule is that the excitations on $\mathcal{M}$ is the same as bulk excitations. First, in the module category language, we can define bulk excitations by $\mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$, namely, view $\mathcal{C}$ itself as $\mathcal{C}$-$\mathcal{C}$-bimodule, the "trivial defect", and the bulk excitations are naturally excitations on the trivial defect. It is straightforward to verify that the data of a functor $F \in \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$ is fully encoded in the object $F(\mathbf{1}) \in \mathcal{C}$, and the natural isomorphisms

$$F(\mathbf{1}) \otimes A \cong F(\mathbf{1} \otimes A) \cong F(A) \cong F(A \otimes \mathbf{1}) \cong A \otimes F(\mathbf{1}). \tag{98}$$

$\mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$ is equivalent to the Drinfeld center $Z(\mathcal{C})$, and the above natural isomorphisms are just the half-braidings. The notions of braiding and Drinfeld center will be introduced later.

Second, to see that $\mathrm{Fun}_{\mathcal{C}^{\vee}_{\mathcal{M}} \boxtimes \mathcal{C}}(\mathcal{M}, \mathcal{M})$ is equivalent to $Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$, note that an object in $\mathrm{Fun}_{\mathcal{C}^{\vee}_{\mathcal{M}} \boxtimes \mathcal{C}}(\mathcal{M}, \mathcal{M})$ can be identified with an object in $\mathcal{C}$, via the following functor that forgets the module functor structure over $\mathcal{C}$

$$\mathrm{Fun}_{\mathcal{C}^{\vee}_{\mathcal{M}} \boxtimes \mathcal{C}}(\mathcal{M}, \mathcal{M}) \xrightarrow{\text{forget}} \mathrm{Fun}_{\mathcal{C}^{\vee}_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}) \simeq \mathcal{C}. \tag{99}$$

Recall that the equivalence is given by

$$\mathcal{C} \simeq \mathrm{Fun}_{\mathcal{C}^\vee_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}), \quad X \mapsto X \otimes -. \tag{100}$$

Moreover, $X \otimes -$ being a left $\mathcal{C}$-module functor means that for any $A \in \mathcal{C}$ and $M \in \mathcal{M}$ there is a natural isomorphism $b^X_{A,M} : X \otimes (A \otimes M) \to A \otimes (X \otimes M)$. Such $b^X_{A,-}$ is a natural transformation between $X \otimes (A \otimes -) \cong (X \otimes A) \otimes -$ and $A \otimes (X \otimes -) \cong (A \otimes X) \otimes -$ viewed as module functors in $\mathrm{Fun}_{\mathcal{C}^\vee_{\mathcal{M}}}(\mathcal{M}, \mathcal{M})$, thus can be identified with an isomorphism $X \otimes A \cong A \otimes X$ in $\mathcal{C}$. It is in fact the half-braiding in (98), where X corresponds to $F(\mathbf{1})$. It is then straightforward to verify $\mathrm{Fun}_{\mathcal{C}^\vee_{\mathcal{M}} \boxtimes \mathcal{C}}(\mathcal{M}, \mathcal{M}) \simeq Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$. Interestingly, this result is symmetric in $\mathcal{C}$ and $\mathcal{C}^\vee_{\mathcal{M}}$, thus $\mathrm{Fun}_{\mathcal{C}^\vee_{\mathcal{M}} \boxtimes \mathcal{C}}(\mathcal{M}, \mathcal{M}) \simeq Z(\mathcal{C}) \simeq Z(\mathcal{C}^\vee_{\mathcal{M}})$. We see that $\mathcal{C}, \mathcal{C}^\vee_{\mathcal{M}}$ share the same category of excitations as the invertible bimodule $\mathcal{M}$. Note that this only established the equivalence between $Z(\mathcal{C})$ and $Z(\mathcal{C}^\vee_{\mathcal{M}})$ as tensor categories. A more careful analysis shows that the above equivalence reverses the half braidings of $Z(\mathcal{C})$ and $Z(\mathcal{C}^\vee_{\mathcal{M}})$, and thus a braided equivalence $\overline{Z(\mathcal{C})} \simeq Z(\mathcal{C}^\vee_{\mathcal{M}})$. Physically, invertible bimodule categories correspond to "transparent" defects that allow excitations to tunnel through freely.

More generally, moving quasiparticles between bulks, boundaries and defects is captured by the following functors. Let $\mathcal{M}$ be a $\mathcal{C}$-module category, and $F \in Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$, F induces a module functor $L^{\mathcal{M}}(F)$. For any $M \in \mathcal{M}, C \in \mathcal{C}$,

$$L^{\mathcal{M}}(F)(M) = F(\mathbf{1}) \otimes M, \tag{101}$$

$$\begin{array}{ccc} L^{\mathcal{M}}(F)(C \otimes M) & \xrightarrow{\quad\beta_{C,M}\quad} & C \otimes L^{\mathcal{M}}(F)(M) \\ \| & & \| \\ F(\mathbf{1}) \otimes (C \otimes M) \longrightarrow (F(\mathbf{1}) \otimes C) \otimes M & \longrightarrow (C \otimes F(\mathbf{1})) \otimes M \longrightarrow & C \otimes (F(\mathbf{1}) \otimes M) \end{array} \tag{102}$$

thus $(L^{\mathcal{M}}(F), \beta)$ is a module functor. And if $\mathcal{M}$ is a $\mathcal{C}$-$\mathcal{D}$-bimodule, $D \in \mathcal{D}$,

$$\begin{array}{ccc} L^{\mathcal{M}}(F)(M \otimes D) & \xrightarrow{\gamma_{M,D}} & L^{\mathcal{M}}(F)(M) \otimes D \\ \| & & \| \\ F(\mathbf{1}) \otimes (M \otimes D) & \longrightarrow & (F(\mathbf{1}) \otimes M) \otimes D \end{array} \tag{103}$$

$(L^{\mathcal{M}}(F), \beta, \gamma)$ is a bimodule functor. Note that $L^{\mathcal{M}}$ is in fact a functor from $Z(\mathcal{C}) = \mathrm{Fun}_{\mathcal{C}|\mathcal{C}}(\mathcal{C}, \mathcal{C})$ to $\mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M})$. There is a similar functor $R^{\mathcal{M}} : Z(\mathcal{D}) = \mathrm{Fun}_{\mathcal{D}|\mathcal{D}}(\mathcal{D}, \mathcal{D}) \to \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M})$. These functors physically describe the excitations condensing to, or tunneling through the boundaries and defects. In particular, when $\mathcal{M}$ is an invertible bimodule, we have $Z(\mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{C}|\mathcal{D}}(\mathcal{M}, \mathcal{M}) \simeq Z(\mathcal{D})$; $L^{\mathcal{M}}, R^{\mathcal{M}}$ are equivalence tensor functors and the composition $(R^{\mathcal{M}})^{-1} L^{\mathcal{M}}$ is even a braided equivalence.

8 Examples of module category

8.1 Semisimple module categories over $\mathrm{Rep}\,\mathbb{Z}_n$

Let $\mathrm{Rep}\,\mathbb{Z}_n$ denote the category of finite dimension representations over $\mathbb{C}$ of the group $\mathbb{Z}_n$. $\mathrm{Rep}\,\mathbb{Z}_n$ is a unitary fusion category. We denote the simple objects of $\mathrm{Rep}\,\mathbb{Z}_n$ by $i \in \{0, 1, 2, \cdots, n-1\}$. The tensor product $i \otimes j = i + j \mod n$; the unit is 0; associativity and unit isomorphisms are identity.

For each factor of n, say $q|n$, $\mathrm{Rep}\,\mathbb{Z}_q$ is a module category over $\mathrm{Rep}\,\mathbb{Z}_n$. To distinguish, denote simple objects of $\mathrm{Rep}\,\mathbb{Z}_q$ by $\overline{0}, \overline{1}, \overline{q-1}$. The tensor product $i \otimes \overline{j} = \overline{i + j \mod q}$; associativity and unit isomorphisms are identity.

Let $\mathcal{M}$ be a semisimple module category over $\mathrm{Rep}\,\mathbb{Z}_n$. Take a simple object $M \in \mathcal{M}$, and denote $iM := i \otimes M$. Define $\mathrm{rank}(iM) =$ number of simple objects in direct sum decomposition of iM, and we see $\mathrm{rank}((i+1)M)) = \mathrm{rank}(1 \otimes iM) \geq \mathrm{rank}(iM)$. However, $M \cong 0M \cong nM$, we conclude that $\mathrm{rank}(iM) = 1$, i.e. iM is simple. Hence the fusion rules of iM are clear: for a factor of n, say $q|n$, there is a set of fusion rules such that $qM \cong M$ and for all $0 < p < q$, $pM \ncong M$. We denote the submodule category $\{0M, 1M, \cdots, (q-1)M\}$ by $\mathcal{M}_q$, and we will show that $\mathcal{M}_q$ with nontrivial associativity and unit isomorphisms is isomorphic to $\mathrm{Rep}\,\mathbb{Z}_q$ as module categories over $\mathrm{Rep}\,\mathbb{Z}_n$.

Let the associativity of $\mathcal{M}_q$ be

$$(i \otimes j) \otimes kM \xrightarrow{\alpha_{i,j,k}} i \otimes (j \otimes kM)\ . \tag{104}$$

In this section we mainly deal with simple objects, therefore we simply regard $\alpha_{i,j,k}$ as a complex number. (We omit $||$ if not confusing)

Consider the pentagon

$$\begin{array}{ccc}
((i \otimes j) \otimes k) \otimes lM & = = = = = = & (i \otimes (j \otimes k)) \otimes lM \\
\downarrow \scriptstyle{\alpha_{i+j,k,l}} & & \downarrow \scriptstyle{\alpha_{i,j+k,l}} \\
(i \otimes j) \otimes (k \otimes lM) & & i \otimes ((j \otimes k) \otimes lM) \\
& \searrow^{\alpha_{i,j,k+l}} \quad {}^{\alpha_{j,k,l}}\swarrow & \\
& i \otimes (j \otimes (k \otimes lM)) &
\end{array} \tag{105}$$

Thus we have

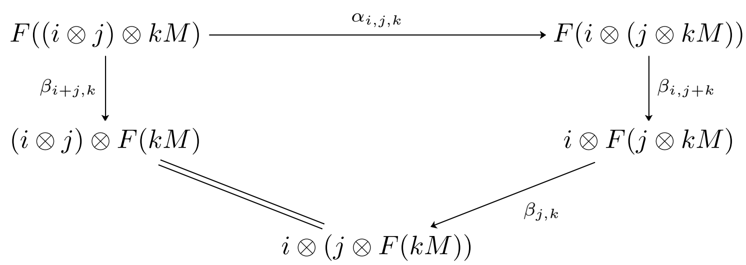
$$\alpha_{i+j,k,l}\alpha_{i,j,k+l} = \alpha_{i,j+k,l}\alpha_{j,k,l}. \tag{106}$$

Set $k = 0$ in eq.(106) we find that $\alpha_{i,0,l} = \alpha_{j,0,l}$. Set $j = 0$ in eq.(106) we find that $\alpha_{i,0,k+l} = \alpha_{0,k,l}$.

Now we define the module functor

$$F : \mathcal{M}_q \to \operatorname{Rep} \mathbb{Z}_q,$$
$$F(iM) = \bar{i}, \quad F(i \otimes jM) \xrightarrow{\beta_{i,j}} i \otimes F(jM)\ .$$

Consider the pentagon



(107)

Thus we have
$$\beta_{i+j,k} = \alpha_{i,j,k}\beta_{i,j+k}\beta_{j,k}. \tag{108}$$
Set $j = 0$ in (108) we find that $\beta_{0,k} = \alpha_{i,0,k}^{-1}$. Set $i = 0$ in (108) we find that $\beta_{0,j+k} = \alpha_{0,j,k}^{-1}$. Thus $\beta_{0,k}$ are uniquely determined by α, with no incompatibility.

Observe that (108) provides us a way to determine $\beta_{i+j,k}$ with $\beta_{i,j+k}, \beta_{j,k}$, which is associative, like some kind of "addition"

$$\begin{array}{llll} & \beta_{i+(j+k),l} & = \alpha_{i,j+k,l}\beta_{i,j+k+l}\beta_{j+k,l} & = \alpha_{i,j+k,l}\alpha_{j,k,l}\beta_{i,j+k+l}\beta_{j,k+l}\beta_{k,l} \\ = & \beta_{(i+j)+k,l} & = \alpha_{i+j,k,l}\beta_{i+j,k+l}\beta_{k,l} & = \alpha_{i+j,k,l}\alpha_{i,j,k+l}\beta_{i,j+k+l}\beta_{j,k+l}\beta_{k,l}, \end{array} \tag{109}$$

Therefore, we can choose nonzero numbers as $\beta_{1,k}$, such that

$$\beta_{0,k} = \beta_{n,k} = \alpha_{1,n-1,k}\alpha_{1,n-2,k}\cdots\alpha_{1,1,k}\beta_{1,k}\beta_{1,k+1}\cdots\beta_{1,k+n-1}, \tag{110}$$

i.e.,

$$(\beta_{1,0}\beta_{1,1}\cdots\beta_{1,q-1})^{n/q} = (\alpha_{1,n-1,0}\alpha_{1,n-2,0}\cdots\alpha_{1,1,0}\alpha_{1,0,0})^{-1}, \tag{111}$$

and other $\beta_{i,k}$ are determined recursively with (108). We can make a choice

$$\beta_{i,k} = \alpha_{1,i-1,k}\alpha_{1,i-2,k}\cdots\alpha_{1,1,k}\beta_{1,k}\beta_{1,k+1}\cdots\beta_{1,k+i-1}. \tag{112}$$

And it is easy to check that (108) is satisfied. Now we finished the definition of the module functor F, and it is obviously an isomorphic functor. We conclude that all indecomposable semisimple module categories over $\operatorname{Rep}\mathbb{Z}_n$ are $\operatorname{Rep}\mathbb{Z}_q$ where $q|n$. When q is a common divisor of n, m, $\operatorname{Rep}\mathbb{Z}_q$ is obviously a $\operatorname{Rep}\mathbb{Z}_n$-$\operatorname{Rep}\mathbb{Z}_m$-bimodule category.

8.2 Module Functors from $\operatorname{Rep}\mathbb{Z}_q$ to $\operatorname{Rep}\mathbb{Z}_p$

Let $\operatorname{Rep}\mathbb{Z}_q$ and $\operatorname{Rep}\mathbb{Z}_p$ be module categories over $\operatorname{Rep}\mathbb{Z}_n$. We investigate module functors between them in this subsection. First we figure out the object maps. Consider

$$\begin{aligned} F : \operatorname{Rep}\mathbb{Z}_q &\to \operatorname{Rep}\mathbb{Z}_p \\ \overline{0} &\mapsto M. \end{aligned}$$

Because $M = F(\overline{0}) = F(\overline{q}) \cong q \otimes F(\overline{0}) = q \otimes M$, in general, M has such form

$$M = \bar{i} \oplus \overline{q+i} \oplus \overline{2q+i} \oplus \cdots \oplus \bar{j} \oplus \overline{q+j} \oplus \overline{2q+j} \oplus \cdots , \tag{113}$$

Let the greatest common divisor of q, p be $(q, p) = r$, we can reduce M to $[i \otimes (\overline{0} \oplus \overline{r} \oplus \overline{2r} \oplus \cdots \oplus \overline{p-r})] \oplus [j \otimes (\overline{0} \oplus \overline{r} \oplus \overline{2r} \oplus \cdots \oplus \overline{p-r})] \oplus \cdots$. Therefore, a simple module functor F from $\operatorname{Rep}\mathbb{Z}_q$ to $\operatorname{Rep}\mathbb{Z}_p$ has the object map as

$$F(\overline{0}) = x \otimes (\overline{0} \oplus \overline{r} \oplus \overline{2r} \oplus \cdots \oplus \overline{p-r}). \tag{114}$$

Let $p = rs, q = rt$ and the least common multiple of q, p be $\langle q, p\rangle = rst = qs$. Then we figure out the possibilities of

$$F(1 \otimes \bar{i}) \xrightarrow{\beta_{1,i}} 1 \otimes F(\bar{i}) \ . \tag{115}$$

Follow the discussion of last subsection we know it suffices to choose all $\beta_{1,i}$ to determine all $\beta_{i,j}$. $\beta_{1,i}$ has s components

$$\beta_{1,i} = \beta_{1,i,0}\, \mathrm{id}_{\overline{x+i+1}} \oplus \beta_{1,i,1}\, \mathrm{id}_{\overline{x+i+1+r}} \oplus \cdots \oplus \beta_{1,i,s-1}\, \mathrm{id}_{\overline{x+i+1+p-r}} . \tag{116}$$

In total we have qs parameters to determine β. Meanwhile, if $F, F' : \operatorname{Rep}\mathbb{Z}_q \to \operatorname{Rep}\mathbb{Z}_p$ have the same object map (114), a natural transformation $\nu : F \to F'$ also has qs parameters

$$\nu_i = \nu_{i,0}\, \mathrm{id}_{\overline{x+i}} \oplus \nu_{i,1}\, \mathrm{id}_{\overline{x+i+r}} \oplus \cdots \oplus \nu_{i,s-1}\, \mathrm{id}_{\overline{x+i+p-r}}, \tag{117}$$

and the diagram (89) is equivalent to the following diagram (take the first component of $F(\overline{0})$)

$$\begin{array}{ccccccc}
F(\overline{0}) = F(\overline{qs}) & & 1 \otimes F(\overline{qs-1}) & \cdots\cdots & (qs-1) \otimes F(\overline{1}) & & qs \otimes F(\overline{0}) \\
\\
\overline{qs+x} & \xrightarrow{\beta_{1,q-1,s-t}} & 1 \otimes \overline{qs+x-1} & \cdots\cdots & (qs-1) \otimes \overline{x+1} & \xrightarrow{\beta_{1,0,0}} & qs \otimes \overline{x} \\
\downarrow{\scriptstyle \nu_{0,0}} & & \downarrow{\scriptstyle \nu_{q-1,s-t}} & & \downarrow{\scriptstyle \nu_{1,0}} & & \downarrow{\scriptstyle \nu_{0,0}} \\
\overline{qs+x} & \xrightarrow{\beta'_{1,q-1,s-t}} & 1 \otimes \overline{qs+x-1} & \cdots\cdots & (qs-1) \otimes \overline{x+1} & \xrightarrow{\beta'_{1,0,0}} & qs \otimes \overline{x} \\
\\
F'(\overline{0}) = F'(\overline{qs}) & & 1 \otimes F'(\overline{qs-1}) & \cdots\cdots & (qs-1) \otimes F'(\overline{1}) & & qs \otimes F'(\overline{0})
\end{array} \tag{118}$$

where a part of the middle looks like

$$
\begin{array}{ccccc}
(q-1)\otimes F(\overline{qs-q+1}) & & q\otimes F(\overline{qs-q}) & & (q+1)\otimes F(\overline{qs-q-1}) \\
 & & & & \\
(q-1)\otimes\overline{qs-q+1+x} & \xrightarrow{\beta_{1,0,s-t}} & q\otimes\overline{qs-q+x} & \xrightarrow{\beta_{1,q-1,s-2t}} & (q+1)\otimes\overline{qs-q-1+x} \\
\downarrow{\scriptstyle \nu_{1,s-t}} & & \downarrow{\scriptstyle \nu_{0,s-t}} & & \downarrow{\scriptstyle \nu_{q-1,s-2t}} \\
(q-1)\otimes\overline{qs-q+1+x} & \xrightarrow{\beta'_{1,0,s-t}} & q\otimes\overline{qs-q+x} & \xrightarrow{\beta'_{1,q-1,s-2t}} & (q+1)\otimes\overline{qs-q-1+x} \\
 & & & & \\
(q-1)\otimes F'(\overline{qs-q+1}) & & q\otimes F'(\overline{qs-q}) & & (q+1)\otimes F'(\overline{qs-q-1})
\end{array}
\tag{119}
$$

In general the second index of β and the first index of ν decrease by 1 each block, and the last index of β, ν remain the same among q blocks and decrease by t when entering the next q blocks. Thus we see the diagram (118) encode all information of β, ν.

Now we can put β into canonical forms. First we can make all $\beta_{1,i,j}$ equal to each other by choosing appropriate ν; second, when $\beta_{1,i,j} = \beta$

$$
F(1\otimes\bar{i}) \xrightarrow{\beta\,\mathrm{id}} 1\otimes F(\bar{i})\ . \tag{120}
$$

Thus $1 = \beta_{0,0} = \beta_{n,0} = \beta_{1,n-1}\beta_{1,n-2}\cdots\beta_{1,0} = \beta^n$, i.e. β is n-th root of 1. Third, from (118) we see that $\beta^{qs} = (\beta')^{qs}$, i.e. two module functors F, F' are isomorphic if β, β' differ by qs-th root of 1, or $\langle q,p\rangle$-th root of 1. Therefore we have canonical forms of all the simple objects in $\mathrm{Fun}_{\mathrm{Rep}\,\mathbb{Z}_n}(\mathrm{Rep}\,\mathbb{Z}_q, \mathrm{Rep}\,\mathbb{Z}_p)$

$$
\begin{gathered}
F_{xy} : \mathrm{Rep}\,\mathbb{Z}_q \to \mathrm{Rep}\,\mathbb{Z}_p, \\
F_{xy}(\overline{0}) = x\otimes(\overline{0}\oplus\overline{r}\oplus\overline{2r}\oplus\cdots\oplus\overline{p-r}), \qquad r=(q,p), \\
F_{xy}(1\otimes\overline{k}) \xrightarrow{\exp\left(2\pi i\frac{y}{n}\right)\,\mathrm{id}} 1\otimes F_{xy}(\overline{k})\ , \\
F_{xy}\cong F_{x'y'} \Longleftrightarrow \begin{cases} x = x' \mod p, \\ y = y' \mod \dfrac{n}{\langle q,p\rangle}. \end{cases}
\end{gathered}
\tag{121}
$$

Let $\mathrm{Rep}\,\mathbb{Z}_q, \mathrm{Rep}\,\mathbb{Z}_p$ be $\mathrm{Rep}\,\mathbb{Z}_n$-$\mathrm{Rep}\,\mathbb{Z}_m$-bimodule categories, i.e. $\langle q,p\rangle | (n,m)$ with all the bimodule structures trivial. Similar to the discussion above, the object map for a bimodule functor $F : \mathrm{Rep}\,\mathbb{Z}_q \to \mathrm{Rep}\,\mathbb{Z}_p$ is as (114). Use the same trick as in diagram (118), we can make

$$
F(1\otimes\bar{i}) \xrightarrow{\beta\,\mathrm{id}} 1\otimes F(\bar{i})\ , \tag{122}
$$

and due to diagram (96) we also require that

$$F(\bar{i}\otimes 1)\xrightarrow{\gamma\,\mathrm{id}} F(\bar{i})\otimes 1\ , \tag{123}$$

where β is an n-th root of 1, and γ is an m-th root of 1. Moreover, we can use natural transformations to shift β,γ together by $\langle q,p\rangle$-th roots of 1, thus sometimes it will be better to use the difference between β,γ to describe F (half-braiding)

$$F(\bar{i})\otimes 1\xrightarrow{\gamma^{-1}\beta\,\mathrm{id}} 1\otimes F(\bar{i}) \tag{124}$$

We have canonical forms of the simple objects in $\mathrm{Fun}_{\mathrm{Rep}\,\mathbb{Z}_n|\,\mathrm{Rep}\,\mathbb{Z}_m}(\mathrm{Rep}\,\mathbb{Z}_q,\mathrm{Rep}\,\mathbb{Z}_p)$:

$$\begin{aligned}
&F_{xyz}:\mathrm{Rep}\,\mathbb{Z}_q\to\mathrm{Rep}\,\mathbb{Z}_p,\\
&F_{xyz}(\bar{0})=x\otimes(\bar{0}\oplus\bar{r}\oplus\overline{2r}\oplus\cdots\oplus\overline{p-r}), \qquad r=(q,p),\\
&F_{xyz}(1\otimes\bar{k})\xrightarrow{\exp\left(2\pi i\frac{y}{n}\right)\mathrm{id}} 1\otimes F_{xyz}(\bar{k})\ ,\\
&F_{xyz}(\bar{k}\otimes 1)\xrightarrow{\exp\left(2\pi i\frac{z}{m}\right)\mathrm{id}} F_{xyz}(\bar{k})\otimes 1\ ,\\
&F_{xyz}(\bar{k})\otimes 1\xrightarrow{\exp\left(2\pi i\frac{ym-zn}{nm}\right)\mathrm{id}} 1\otimes F_{xyz}(\bar{k})\ ,
\end{aligned} \tag{125}$$

$$F_{xyz}\cong F_{x'y'z'}\iff\begin{cases} x=x' \mod p,\\ y=y' \mod \dfrac{n}{\langle q,p\rangle},\\ ym-zn=y'm-z'n \mod nm.\end{cases}$$

The canonical forms of module functors satisfy

$$F_{xyz}F_{x'y'z'}=F_{(x+x')(y+y')(z+z')}. \tag{126}$$

As a special case, $n=m=q=p$, we get the Drinfeld center of $\mathrm{Rep}\,\mathbb{Z}_n$, $Z(\mathrm{Rep}\,\mathbb{Z}_n)=\mathrm{Fun}_{\mathrm{Rep}\,\mathbb{Z}_n|\,\mathrm{Rep}\,\mathbb{Z}_n}(\mathrm{Rep}\,\mathbb{Z}_n,\mathrm{Rep}\,\mathbb{Z}_n)$, whose simple objects are

$$F_{xy}:\mathrm{Rep}\,\mathbb{Z}_n\to\mathrm{Rep}\,\mathbb{Z}_n,$$

$$F_{xy}(\bar{0})=x,\qquad F_{xy}(\bar{k})\otimes 1\xrightarrow{\exp\left(2\pi i\frac{y}{n}\right)\mathrm{id}} 1\otimes F_{xy}(\bar{k})\ , \tag{127}$$

$$F_{xy}\cong F_{x'y'}\iff\begin{cases} x=x' \mod n,\\ y=y' \mod n.\end{cases}$$

8.3 i**Vec as invertible** $\operatorname{Rep}\mathbb{Z}_n$-$\operatorname{Rep}\mathbb{Z}_n$**-bimodule**

As we have discussed above, $\operatorname{Rep}\mathbb{Z}_1 = \mathbf{Vec}$ is a $\operatorname{Rep}\mathbb{Z}_n$-$\operatorname{Rep}\mathbb{Z}_n$-bimodule, which is not invertible. However, there are nontrivial bimodule structures making the category of vector spaces an invertible $\operatorname{Rep}\mathbb{Z}_n$-$\operatorname{Rep}\mathbb{Z}_n$-bimodule. We denote this invertible bimodule by $i\mathbf{Vec}$. We will list the data of $i\mathbf{Vec}$ below without the proof why it is invertible, since we are not going to discuss tensor product of bimodule categories in detail. $\operatorname{Rep}\mathbb{Z}_n$ itself is invertible as bimodule, and the defects described by $\operatorname{Rep}\mathbb{Z}_n$ and $i\mathbf{Vec}$ allow excitations tunneling freely.

$i\mathbf{Vec}$ has only one simple object $\mathbb{C}$, the nontrivial bimodule structure is

$$(1 \otimes \mathbb{C}) \otimes 1 \xrightarrow{\exp\left(\frac{2\pi i}{n}\right)} 1 \otimes (\mathbb{C} \otimes 1) \ . \tag{128}$$

We can write down $\operatorname{Fun}_{\operatorname{Rep}\mathbb{Z}_n|\operatorname{Rep}\mathbb{Z}_n}(i\mathbf{Vec}, i\mathbf{Vec})$:

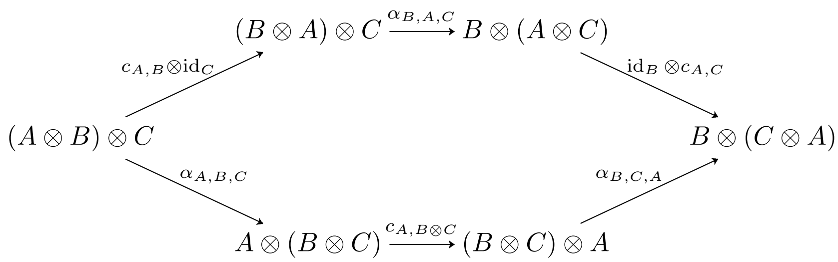
$$\begin{gathered} F_{yz} : i\mathbf{Vec} \to i\mathbf{Vec}, \\ F_{yz}(\mathbb{C}) = \mathbb{C}, \\ F_{yz}(1 \otimes \mathbb{C}) \xrightarrow{\exp\left(2\pi i \frac{y}{n}\right) \mathrm{id}} 1 \otimes F_{yz}(\mathbb{C}) \ , \\ F_{yz}(\mathbb{C} \otimes 1) \xrightarrow{\exp\left(2\pi i \frac{z}{n}\right) \mathrm{id}} F_{yz}(\mathbb{C}) \otimes 1 \ , \\ F_{yz} \cong F_{y'z'} \iff \begin{cases} y = y' & \bmod n, \\ z = z' & \bmod n. \end{cases} \end{gathered} \tag{129}$$

It is interesting to check the result of $L^{i\mathbf{Vec}}, R^{i\mathbf{Vec}}$. We see $L^{i\mathbf{Vec}}(F_{xy}) = F_{yx}$ and $R^{i\mathbf{Vec}}(F_{xy}) = F_{x,-y}$. Both $L^{i\mathbf{Vec}}$ and $R^{i\mathbf{Vec}}$ are equivalence functors. Physically, a bulk excitation F_{xy} can tunnel through the defect $i\mathbf{Vec}$ to the other side, by the action of $(R^{i\mathbf{Vec}})^{-1}L^{i\mathbf{Vec}}$ and become another type of bulk excitation $F_{y,-x}$.

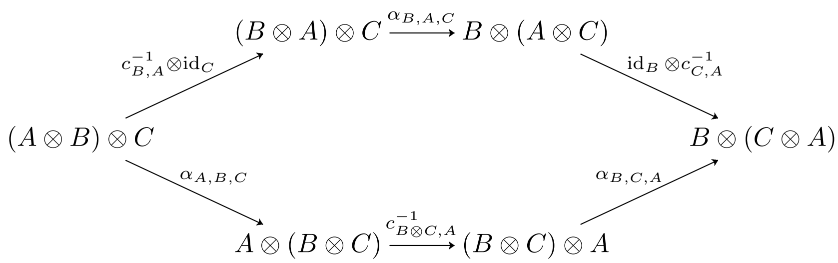
9 Unitary Braided Fusion Category

We have discussed unitary fusion category and module category in the previous sections, which can describe the wavefunctions of string-net model and its gapped boundary. In this section we add the *braiding* structure and introduce unitary braided fusion category (UBFC), which describes the (non-Abelian) statistics of quasiparticle excitations or simply *anyons*.

Definition 9.1 (Unitary braided fusion category)**.** A unitary fusion category is called braided if there are natural isomorphisms: $c_{A,B} : A \otimes B \to B \otimes A$, called braidings, satisfying $c_{A,B}^{-1} = c_{A,B}^{\dagger}$ and the hexagon equations:



(130)



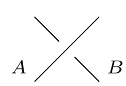
(131)

We denote by $\overline{\mathcal{C}}$ the same tensor category as $\mathcal{C}$ but with reversed braiding $\overline{c}_{A,B} := c_{B,A}^{-1}$.

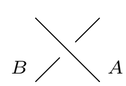
Exercise 9.1. Prove

$$c_{\mathbf{1},A}\lambda_A = \rho_A, \quad c_{A,\mathbf{1}}\rho_A = \lambda_A. \tag{132}$$

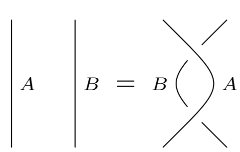
Graphically, the braiding $c_{A,B}$ is drawn as

$c_{A,B} =$  , (133)

and its inverse as

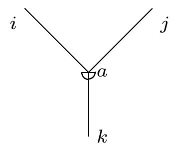
$c_{A,B}^{-1} =$  , (134)

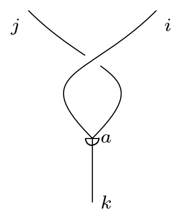
which satisfy

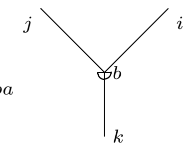
 . (135)

In terms of simple objects, the associativity isomorphism $\alpha_{i,j,k}$ can be represented by the F-matrix. The braiding can be similarly represented by the

R-matrix, which is also unitary:

$c_{i,j}$ 

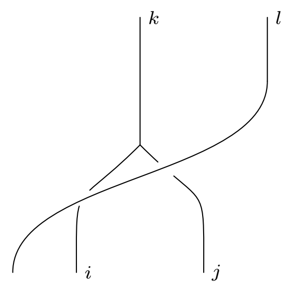
$=$ 

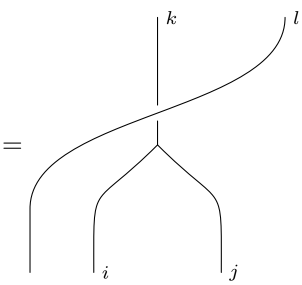
$= \sum_b R^{ij}_{k,ba}$ 

, (136)

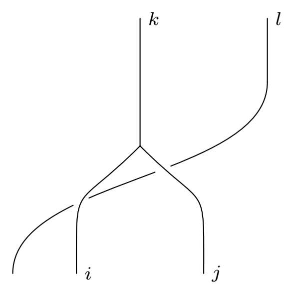
$$R^{ij}_{k,ba} = (c_{i,j})_{k,ba} = |p^{k,b}_{ji} c_{i,j} (p^{k,a}_{ij})^\dagger|. \tag{137}$$

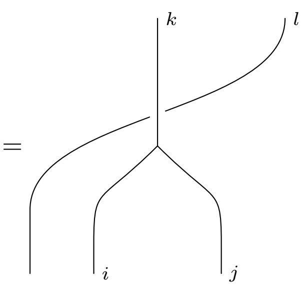
As long as N^{ij}_k, F^{ijk}_l and R^{ij}_k are known, any graphs can be calculated. One important trick is that, since $c_{i,j}$ are natural isomorphisms, a string can slide above or under any morphisms, in particular the vertices:



$=$ 

, (138)

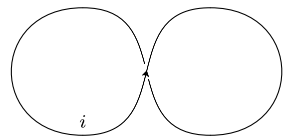


$=$ 

. (139)

One can say that the data $(N^{ij}_k, F^{ijk}_l, R^{ij}_k)$ represents a unitary braided fusion category. The definitions above can be translated into algebraic equations of $(N^{ij}_k, F^{ijk}_l, R^{ij}_k)$. However, solving these equations can be quite tedious. On the other hand, F^{ijk}_l and R^{ij}_k depend on the choice of basis for the vertices, which makes it difficult to recognize equivalent solutions. Next we introduce some basis independent quantities that characterize a unitary braided fusion category.

Definition 9.2 (Topological spin, T matrix)**.** The topological spin θ_i of a simple object i is given by

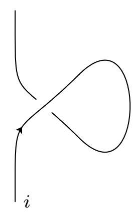
$$\theta_i = \frac{1}{d_i} \operatorname{tr} c_{i,i} = \frac{1}{d_i}$$  . (140)

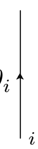
The corresponding T matrix is a diagonal matrix indexed by simple objects whose diagonal elements are the topological spins:

$$T_{ij} = \theta_i \delta_{ij}. \tag{141}$$

Exercise 9.2. Prove the followings properties of θ_i:

$$\theta_i = \sum_j \frac{d_j}{d_i} \operatorname{Tr} R^{ii}_j, \tag{142}$$

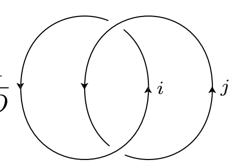


$= \theta_i$  , (143)

$$\theta_i \overline{\theta_i} = 1, \tag{144}$$

$$\theta_a = \theta_{a^*}. \tag{145}$$

Definition 9.3 (S matrix)**.** S matrix, also indexed by simple objects, is given by

$$S_{ij} = \frac{1}{D} \operatorname{tr} c_{j^*,i} c_{i,j^*} = \frac{1}{D}$$  , (146)

where $D = \dim(\mathcal{C}) = \sqrt{\sum_i d_i^2}$ is called the total (or global) quantum dimension.

The total quantum dimension is a measure of the size of fusion category.

Lemma 9.1 (EO [9])**.** By a fusion subcategory we mean a full subcategory that is a fusion category itself. If $\mathcal{B}$ is a fusion subcategory of $\mathcal{C}$ (or there is a fully faithful tensor embedding $\mathcal{B} \hookrightarrow \mathcal{C}$) then $\dim(\mathcal{B}) \leq \dim(\mathcal{C})$, and the equality holds if and only if $\mathcal{B} \simeq \mathcal{C}$.

Exercise 9.3. Prove the following properties of the S matrix:

$$S_{ij} = \sum_k \frac{d_k}{D} \operatorname{Tr}(R_k^{j^* i} R_k^{ij^*}) = \sum_k \frac{d_k}{D} \frac{\theta_k}{\theta_i \theta_j} N_k^{ij^*}, \tag{147}$$

$$S_{ij} = S_{ji} = \overline{S_{ij^*}}, \tag{148}$$

$$\frac{D}{d_x} S_{ix} S_{jx} = \sum_k N_k^{ij} S_{kx}. \tag{149}$$

It is of particular interest whether the mutual statistics of two anyons is trivial or not, namely

Definition 9.4. The objects X, Y in a UBFC $\mathcal{C}$ are said to *centralize* each other (mutually trivial) if

$$c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y}, \tag{150}$$

where $c_{X,Y} : X \otimes Y \cong Y \otimes X$ is the braiding in $\mathcal{C}$. Equivalently, i, j centralize each other if $S_{ij} = d_i d_j / D$. Given a fusion subcategory $\mathcal{D} \subset \mathcal{C}$, its *centralizer* $\mathcal{D}'|_{\mathcal{C}}$ in $\mathcal{C}$ is the full subcategory of objects in $\mathcal{C}$ that centralize all the objects in $\mathcal{D}$. The centralizer is a fusion subcategory. In particular, $\mathcal{C}'|_{\mathcal{C}}$ is called the Müger center of $\mathcal{C}$.

In a 2+1D topological order (intrinsic without symmetry), we expect that the type of quasiparticles can be measured by the braidings. This is captured by the following notion.

Definition 9.5. A UBFC $\mathcal{C}$ is a unitary modular tensor category (UMTC) if $\mathcal{C}'|_{\mathcal{C}} = \mathbf{Vec}$. Equivalently, its S matrix is invertible.

Remark 9. The T, S matrices of a UMTC form a projective representation of $SL(2, \mathbb{Z})$ (modular transformation),

$$S^4 = I, \quad (ST)^3 = \mathrm{e}^{2\pi \mathrm{i} \frac{c}{8}} S^2, \tag{151}$$

where c is known as the chiral central charge,

$$\mathrm{e}^{2\pi \mathrm{i} \frac{c}{8}} = \sum_i \frac{d_i^2 \theta_i}{D}. \tag{152}$$

Definition 9.6. The Drinfeld center $Z(\mathcal{A})$ of a tensor category $\mathcal{A}$ is a braided tensor category with objects as pairs $(X \in \mathcal{A}, b_{X,-})$, where $b_{X,-} : X \otimes - \to - \otimes X$ are half-braidings that satisfy similar conditions as braidings. Morphisms are those that commute with half-braidings. The tensor product is given by

$$(X, b_{X,-}) \otimes (Y, b_{Y,-}) = (X \otimes Y, (b_{X,-} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes b_{Y,-})), \tag{153}$$

where the associators have been omitted. The braiding is $c_{(X,b_{X,-}),(Y,b_{Y,-})} = b_{X,Y}$.

It is known that $Z(\mathcal{A})$ is a UMTC if $\mathcal{A}$ is a unitary fusion category [16]. Physically, a string-net model constructed from a UFC $\mathcal{A}$ describes a topological order whose quasiparticle excitations are $Z(\mathcal{A})$.

Lemma 9.2 (DGNO [7])**.** Let $\mathcal{D}$ be a fusion subcategory of a UMTC $\mathcal{C}$, then

$$(\mathcal{D}'|_{\mathcal{C}})'|_{\mathcal{C}} = \mathcal{D}, \quad \dim(\mathcal{D})\dim(\mathcal{D}'|_{\mathcal{C}}) = \dim(\mathcal{C}).$$

Definition 9.7. A UBFC $\mathcal{E}$ is a *symmetric* fusion category if $\mathcal{E}'|_{\mathcal{E}} = \mathcal{E}$.

UMTC and symmetric fusion category correspond to two extreme cases, i.e., braiding is non-degenerate and maximally degenerate, respectively. Symmetric fusion categories are closely related to bosonic and fermionic symmetry groups, according to the following theorem

Theorem 9.3 (Deligne [5])**.** A symmetric fusion category is braided equivalent to $\mathrm{Rep}(G, z)$, where G is a finite group, and $z \in G$ is a central element such that $z^2 = 1$, and $\mathrm{Rep}(G, z)$ is the fusion category $\mathrm{Rep}(G)$ equipped with braiding c^z:

$$c^z_{X,Y}(x \otimes_{\mathbb{C}} y) = (-1)^{mn} y \otimes_{\mathbb{C}} x, \quad \forall x \in X, y \in Y, \quad zx = (-1)^m x, zy = (-1)^n y.$$

When $z = 1$ it is $\mathrm{Rep}(G)$ with the usual braiding $x \otimes_{\mathbb{C}} y \to y \otimes_{\mathbb{C}} x$. When $z \neq 1$ it is the fermion number parity. Fermions braid with each other with an extra -1. We introduce $\mathrm{sRep}(G^f) = \mathrm{Rep}(G, z)$ for $z \neq 1$ to emphasize its fermionic nature.

Example 9.1. $\mathrm{sRep}(\mathbb{Z}_2^f)$ is the category of super Hilbert spaces, **sHilb**, that is, $\mathbb{Z}_2$-graded Hilbert spaces with $\mathbb{Z}_2$-graded braiding. It corresponds to invertible fermionic phases with no other symmetries.

In a physical system with no topological order but with global symmetry G, its excitations carries representations of G. Due to Theorem 9.3 we can simply use the symmetric fusion category $\mathcal{E} = \mathrm{Rep}(G, z)$ instead of the symmetry group G to refer to the symmetry. What if a 2+1D system has both symmetry $\mathcal{E}$ and topological order? It is natural to expect that the excitations are still described by a UBFC $\mathcal{C}$. Clearly, excitations carrying symmetry representations should still be there, namely, $\mathcal{E}$ should be a subcategory of $\mathcal{C}$. If the type of an excitation cannot be measured via braidings, it must be measurable by the symmetry. These considerations lead to the following notions.

Definition 9.8. A pair $(\mathcal{C}, \iota)$, a UBFC $\mathcal{C}$ with a fully faithful embedding $\iota : \mathcal{E} \hookrightarrow \mathcal{C}'|_{\mathcal{C}}$, is a UBFC *over* $\mathcal{E}$. Moreover, $\mathcal{C}$ is said a non-degenerate UBFC over $\mathcal{E}$, or $\mathrm{UMTC}_{/\mathcal{E}}$, if $\mathcal{C}'|_{\mathcal{C}} = \mathcal{E}$. Two UBFCs over $\mathcal{E}$, $(\mathcal{C}_1, \iota_1)$ and $(\mathcal{C}_2, \iota_2)$ are equivalent if there is a braided monoidal equivalence $F : \mathcal{C}_1 \to \mathcal{C}_2$ such that $F\iota_1 = \iota_2$.

The excitations in a 2+1D topological order with symmetry $\mathcal{E}$ thus should form a $\mathrm{UMTC}_{/\mathcal{E}}$. We recover the usual definition of UMTC when $\mathcal{E}$ is trivial. In this case the subscript is omitted.

However, it is possible the symmetry is somewhat "anomalous". To avoid this, we require that $\mathcal{E}$ can be measured if we use the braiding with some additional particles (physically they are "gauged symmetry defects").

Definition 9.9. Given a $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, its (minimal) *modular extension* is a pair $(\mathcal{M}, \iota_\mathcal{M})$, a UMTC $\mathcal{M}$, together with a fully faithful embedding $\iota_\mathcal{M} : \mathcal{C} \hookrightarrow \mathcal{M}$, such that $\mathcal{E}'|_\mathcal{M} = \mathcal{C}$. Two modular extensions $(\mathcal{M}_1, \iota_{\mathcal{M}_1}), (\mathcal{M}_2, \iota_{\mathcal{M}_2})$ are equivalent if there is a braided monoidal equivalence $F : \mathcal{M}_1 \to \mathcal{M}_2$ such that $F\iota_{\mathcal{M}_1} = \iota_{\mathcal{M}_2}$. We denote the set of equivalence classes of modular extensions of $\mathcal{C}$ by $\mathcal{M}_{ext}(\mathcal{C})$.

Remark 10. Here the condition $\mathcal{E}'|_\mathcal{M} = \mathcal{C}$ is equivalent to $\mathcal{C}'|_\mathcal{M} = \mathcal{E}$, or $\dim(\mathcal{M}) = \dim(\mathcal{C})\dim(\mathcal{E})$. Physically this means that the extra excitations in $\mathcal{M}$ but not in $\mathcal{C}$ all have non-trivial mutual statistics with at least one excitation in $\mathcal{E}$. In fact, let $\mathcal{M}$ be a UMTC that contains a symmetric fusion category $\mathcal{E}$ as a full subcategory, and $\mathcal{D} = \mathcal{E}'|_\mathcal{M}$. Then, $\mathcal{E}$ is a full subcategory of $\mathcal{D}$ ($\mathcal{E}$ centralizes itself) and $\mathcal{D}'|_\mathcal{M} = (\mathcal{E}'|_\mathcal{M})'|_\mathcal{M} = \mathcal{E}$. We see that $\mathcal{D}'|_\mathcal{D} = \mathcal{D} \cap (\mathcal{D}'|_\mathcal{M}) = \mathcal{D} \cap \mathcal{E} = \mathcal{E}$. This means that $\mathcal{D} = \mathcal{E}'|_\mathcal{M}$ is automatically a $\mathrm{UMTC}_{/\mathcal{E}}$, and $\mathcal{M}$ is its modular extension. This will be a useful way to construct $\mathrm{UMTC}_{/\mathcal{E}}$'s from UMTCs.

Remark 11. For a given $\mathrm{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, it is possible that there is no minimal modular extension of $\mathcal{C}$. Such $\mathcal{C}$ has *anomalous* symmetry and can only be realized on the surface of some 3+1D SPT phase. An example was constructed by Drinfeld [6]. It is a $\mathrm{UMTC}_{/\,\mathrm{Rep}(\mathbb{Z}_2\times\mathbb{Z}_2)}$ with rank 5 and $D^2 = 8$. The same example is also discussed in Ref. [2].

To conclude, $\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}$ where $\mathcal{E}'|_\mathcal{M} = \mathcal{C}$ and $\mathcal{M}$ is a UMTC, charaterize a 2+1D topological phase with symmetry, or symmetry enriched topological (SET) phase. This characterization becomes complete after we include a chiral central charge c to address the invertible phases (i.e., those without topological excitations). It is important to note that counting modular extensions of a fixed $\mathcal{C}$ is different from counting topological phases.

Definition 9.10. Two topological phases with symmetry $\mathcal{E}$, labeled by $(\mathcal{E} \xrightarrow{\iota_1} \mathcal{C}_1 \xrightarrow{\iota_{\mathcal{M}_1}} \mathcal{M}_1, c_1)$ and $(\mathcal{E} \xrightarrow{\iota_2} \mathcal{C}_2 \xrightarrow{\iota_{\mathcal{M}_2}} \mathcal{M}_2, c_2)$, are equivalent if $c_1 = c_2$ and there are braided monoidal equivalences $F_\mathcal{C} : \mathcal{C}_1 \to \mathcal{C}_2$, $F_\mathcal{M} : \mathcal{M}_1 \to \mathcal{M}_2$ such that $F_\mathcal{C}\iota_1 = \iota_2$, $\iota_{\mathcal{M}_2} F_\mathcal{C} = F_\mathcal{M}\iota_{\mathcal{M}_1}$.

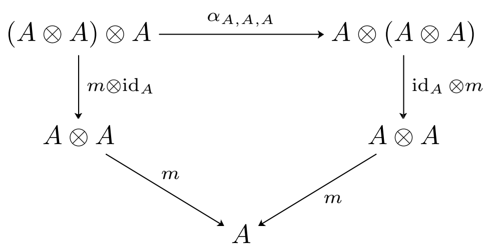
Physically, when counting topological phases, we allow "relabelling" anyons in $\mathcal{C}$ and $\mathcal{M}$ together in a compatible way. But we do not allow mixing "excitations" (anyons in $\mathcal{C}$) with "gauged symmetry defects" (anyons not in $\mathcal{C}$). Also we do not allow "relabelling" local excitations in $\mathcal{E}$, as they are related to the symmetry group which has absolute meaning. For example spin-flip $\mathbb{Z}_2$ can not be considered as the same as layer-exchange $\mathbb{Z}_2$, nor can their representations be relabelled. On the other hand, when counting modular extensions, we fix all the excitations in $\mathcal{C}$ and only allow "relabelling" "gauged symmetry defects" (anyons in $\mathcal{M}$ but not in $\mathcal{C}$).

The embeddings $\iota, \iota_\mathcal{M}$ are important data. However, in Section 11, the embeddings are naturally defined, as we construct $\mathcal{E}, \mathcal{C}$ as full subcategories of $\mathcal{M}$. So we may omit the embeddings to simplify notations whenever there is no ambiguity.

10 Algebra in tensor categories

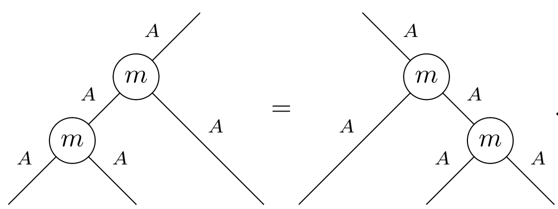
The notion of algebra in a generic tensor category is a generalization of the usual associative algebra, which is simply algebra in the tensor category of vector spaces.

Definition 10.1 (Algebra)**.** An algebra in a tensor category $\mathcal{C}$ is a pair (A, m), an object A with a multiplication morphism $m : A \otimes A \to A$, such that the multiplication is associative:



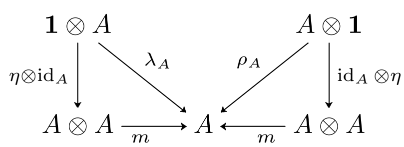
(154)

or compactly, $m(\mathrm{id}_A \otimes m)\alpha_{A,A,A} = m(m \otimes \mathrm{id}_A)$. Graphically,



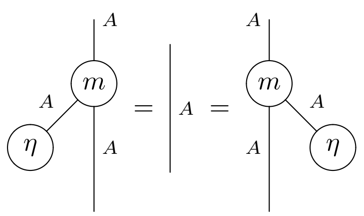
(155)

The algebra is said unital if there is a unital morphism $\eta : \mathbf{1} \to A$ such that



(156)

or compactly, $m(\eta \otimes \mathrm{id}_A) = \lambda_A, m(\mathrm{id}_A \otimes \eta) = \rho_A$. Graphically,



(157)

An algebra morphism between algebras $(A, m), (A', m')$ is a morphism f :

$A \to A'$ that commutes with the multiplications m, m',

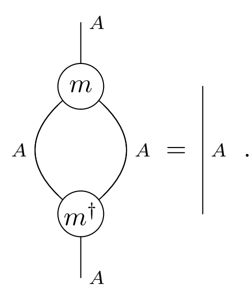
$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\ \downarrow m & & \downarrow m' \\ A & \xrightarrow{f} & A' \end{array} \tag{158}$$

or $fm = m'(f \otimes f)$.

Now assume that $\mathcal{C}$ is a unitary (braided) fusion category, there are more properties of an algebra in $\mathcal{C}$ we will be interested in

Definition 10.2. A unital algebra (A, m, η) in a UFC $\mathcal{C}$ is said connected if $\mathrm{Hom}(\mathbf{1}, A) = \mathbb{C}$.

Definition 10.3. An algebra (A, m) in a UFC $\mathcal{C}$ is said isometric if $mm^\dagger = \mathrm{id}_A$. Graphically,

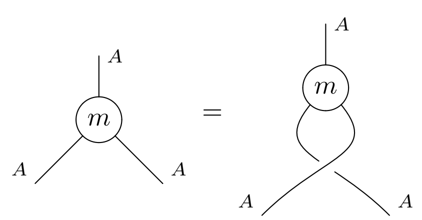


(159)

Definition 10.4. An algebra (A, m) in a UBFC $\mathcal{C}$ is said commutative if

$$\begin{array}{ccc} A \otimes A & \xrightarrow{c_{A,A}} & A \otimes A \\ & {}_{m}\searrow \quad \swarrow_{m} & \\ & A & \end{array} \tag{160}$$

namely $mc_{A,A} = m$. Graphically,



(161)

Definition 10.5 (Condensable algebra)**.** An algebra in a unitary braided fusion category is condensable if it is unital, connected, isometric and commutative.

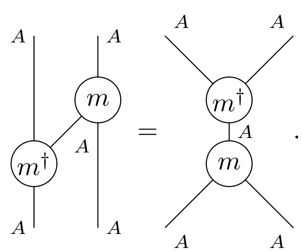
Remark 12. This is an important notion that is widely studied. In the subfactor context it is called (irreducible local) *Q-system* [15]. In category literature it is also known as connected étale algebra (connected commutative separable algebra) [7, 4], or commutative special symmetric C^*-Frobenius algebra [16, 10]. The latter two are more general; they do not require the category to be unitary. In the unitary case, they are equivalent notions [15]. We follow Ref. [13] to call "condensable algebra" for its physical meaning and also simplicity.

The following theorem supports the equivalence of these notions:

Theorem 10.1. Let (A, m, η) be a unital isometric algebra in a unitary fusion category, it satisfies the following *Frobenius condition*:

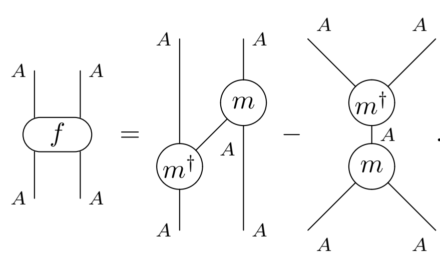
$$(\mathrm{id}_A \otimes m)\alpha_{A,A,A}(m^\dagger \otimes \mathrm{id}_A) = m^\dagger m = (m \otimes \mathrm{id}_A)\alpha^{-1}_{A,A,A}(\mathrm{id}_A \otimes m^\dagger). \tag{162}$$

Proof. We prove the condition graphically, where the associator $\alpha_{A,A,A}$ is implicit. It suffices to check that



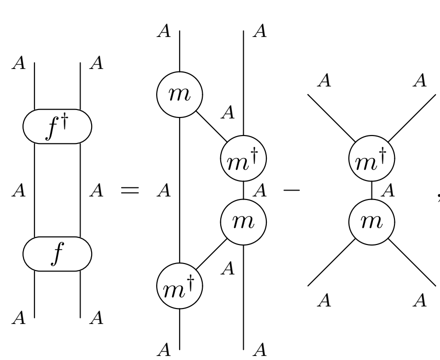
(163)

Let f be their difference:



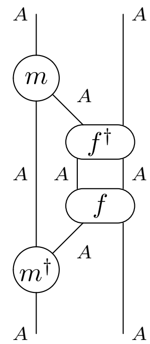
(164)

By direct calculation (applying the associative and isometric properties),



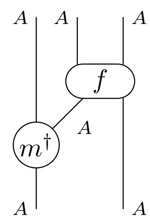
(165)

and



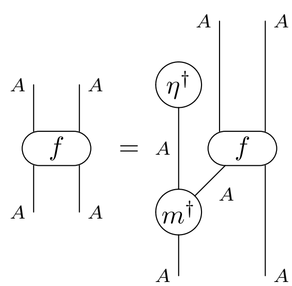
$$= 0. \tag{166}$$

Then, by Corollary 6.6, we know that



$$= 0, \tag{167}$$

which then implies

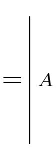


$$= 0. \tag{168}$$

□

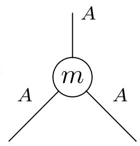
Corollary 10.2. A unital isometric algebra (A, m, η) is self dual, with $b_A = m^\dagger \eta$, $e_A = \eta^\dagger m = b_A^\dagger$, and thus $\dim A = |b_A^\dagger b_A| = |\eta^\dagger \eta|$.

In a unitary fusion category, we can express the data of an algebra A in terms of simple objects. First take a decomposition of A as in (29)

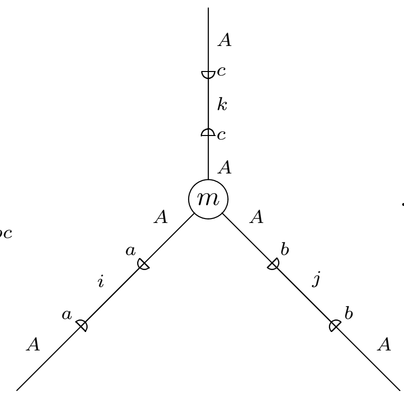
$$\mathrm{id}_A =$$  $$= \sum_i \sum_{a=1}^{N_i^A}$$  $$= \sum_i \sum_{a=1}^{N_i^A} q_{i,a}^A p_A^{i,a}. \tag{169}$$

The multiplication morphism m is then

$m =$

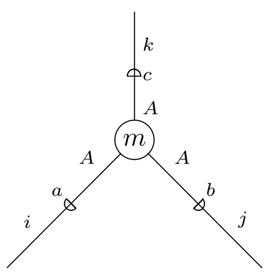


$= \sum_{ijk,abc}$

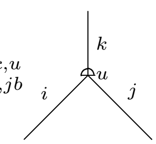


. (170)

The central part can be expressed in terms of basis vertices $p_{ij}^{k,u}$



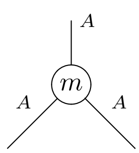
$= \sum_u M_{ia,jb}^{kc,u}$



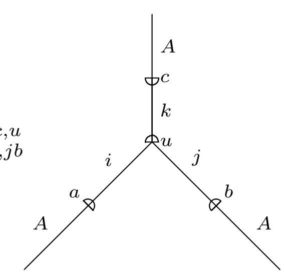
$= \sum_u M_{ia,jb}^{kc,u} p_{ij}^{k,u}.$ (171)

Thus

$m =$



$= \sum_{ijk,abc,u} M_{ia,jb}^{kc,u}$



$= \sum_{ijk,abc,u} M_{ia,jb}^{kc,u} q_{k,c}^{A} p_{ij}^{k,u} \left(p_A^{i,a} \otimes p_A^{j,b}\right).$

(172)

$M_{ia,jb}^{kc,u}$ is the "structure coefficients" of the algebra. In the category of vector spaces **Vec**, the object labels i, j, k and the vertex label u reduce to trivial, and $M_{ia,jb}^{kc,u}$ reduces to structure coefficients of usual associative algebra, with a, b, c the labels of basis vectors.

Exercise 10.1. Show that the associative condition $m(\mathrm{id}_A \otimes m)\alpha_{A,A,A} = m(m \otimes \mathrm{id}_A)$ is equivalent to

$$\sum_w M_{ia,jb}^{rw,u} M_{rw,kc}^{ld,v} = \sum_{sxyz} F_{l,sxy,ruv}^{ijk} M_{jb,kc}^{sz,x} M_{ia,sz}^{ld,y}. \tag{173}$$

Exercise 10.2. The tensor unit $\mathbf{1}$ is simple in a unitary fusion category, thus the unital morphism $\eta : \mathbf{1} \to A$ can be expressed in terms of embeddings $\eta = \eta_a q^A_{\mathbf{1},a}$. If we take λ_i and ρ_i as our basis vertices ($p^{i,1}_{\mathbf{1}i} = \lambda_i, p^{i,1}_{i\mathbf{1}} = \rho_i$), show that unital condition is equivalent

$$\sum_a \eta_a M^{kc,1}_{\mathbf{1}a,jb} = \sum_a \eta_a M^{kc,1}_{jb,\mathbf{1}a} = \delta_{jk}\delta_{bc}. \tag{174}$$

And the connected condition means that the range of index a in the above expressions is 1, thus

$$M^{kc,1}_{\mathbf{1}1,jb} = M^{kc,1}_{jb,\mathbf{1}1} = \eta_1^{-1}\delta_{jk}\delta_{bc}. \tag{175}$$

Exercise 10.3. Show that the isometric condition is equivalent to

$$\sum_{iajbu} M^{kc,u}_{ia,jb}\overline{M^{kc',u}_{ia,jb}} = \delta_{cc'}. \tag{176}$$

Exercise 10.4. Show that the commutative condition is equivalent to

$$M^{kc,u}_{ia,jb} = \sum_v R^{ij}_{k,vu} M^{kc,v}_{jb,ia}. \tag{177}$$

Exercise 10.5. Show that in the category of vector spaces, the above conditions reduce to the usual ones of structure coefficients for an algebra to be associative, unital, commutative, etc.

Next we introduce some basic notions to understand the representation theory of the algebras in a tensor category, which is directly related to the theory of anyon condensation. However, discussing the representation theory in detail is beyond our scope, and we will only quote some results from relevant works which are used later.

Definition 10.6 (Module over an algebra)**.** A (right) *module* over a unital algebra (A, m, η) in $\mathcal{C}$ is a pair (X, ρ), and object $X \in \mathcal{C}$, with an action morphism $\rho : X \otimes A \to X$ satisfying

$$\rho(\rho \otimes \mathrm{id}_A) = \rho(\mathrm{id}_M \otimes m), \tag{178}$$

$$\rho(\mathrm{id}_M \otimes \eta) = \mathrm{id}_M\,. \tag{179}$$

When (A, m, η) is a condensable algebra, we call a module (X, ρ) *local* if

$$\rho c_{A,M} c_{M,A} = \rho. \tag{180}$$

A morphism between modules $(X, \rho), (Y, \tau)$ is a morphism $f : X \to Y$ that commutes with the actions ρ, τ

$$f\rho = \tau(f \otimes \mathrm{id}_A). \tag{181}$$

Remark 13. The module is like "half an algebra". It is similar to express the data of a module in terms of some "structure coefficients", which is left for interested readers.

We denote the category of right A-modules by $\mathcal{C}_A$. A right module (X, ρ) is turned into a left module via the braiding, $(X, \rho c_{X,A}^{-1})$ or $(X, \rho c_{A,X})$, and thus a A-A-bimodule. The relative tensor functor $\otimes_A$ of bimodules then turns $\mathcal{C}_A$ into a fusion category. In general there can be two monoidal structures on $\mathcal{C}_A$, since there are two ways to turn a right module into a bimodule (usually we pick one for definiteness when considering $\mathcal{C}_A$ as a fusion category). The two monoidal structures coincide for the fusion subcategory $\mathcal{C}_A^0$ of local A-modules. Moreover, $\mathcal{C}_A^0$ inherited the braiding from $\mathcal{C}$ and is also a unitary braided fusion category.

Physically, $\mathcal{C}$ describes the excitations in the original bulk phase, $\mathcal{C}_A^0$ describes the excitations in the topological phase after condensing A, and $\mathcal{C}_A$ describes the excitations on the corresponding domain wall between $\mathcal{C}$ and $\mathcal{C}_A^0$. We have explained a similar picture in Section 7, when the bulk can be defined by string-net model of some UFC $\mathcal{A}$, namely $\mathcal{C} \simeq Z(\mathcal{A})$. In this special case, one can find an algebra A in $\mathcal{C} \simeq Z(\mathcal{A})$ such that there are correspondences between $\mathcal{C}_A, \mathcal{C}_A^0$ and UFCs $\mathcal{A}, \mathcal{B}$, $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{M}$ which describe the string-net wavefunctions:

$$Z(\mathcal{A})_A^0 \simeq Z(\mathcal{B}), Z(\mathcal{A})_A \simeq \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}(\mathcal{M}, \mathcal{M}). \tag{182}$$

The following lemma tells us the size of $\mathcal{C}_A$ and $\mathcal{C}_A^0$.

Lemma 10.3 (DMNO [4])**.** Let A be a condensable algebra in UBFC $\mathcal{C}$,

$$\dim(\mathcal{C}_A) = \frac{\dim(\mathcal{C})}{\dim(A)}. \tag{183}$$

If $\mathcal{C}$ is a UMTC, then so is $\mathcal{C}_A^0$, and

$$\dim(\mathcal{C}_A^0) = \frac{\dim(\mathcal{C})}{\dim(A)^2}. \tag{184}$$

11 Stacking of topological phases

We can *stack* two existing topological phases to obtain a third phase, which is better visualized in (2+1)D by just constructing a two-layer system. The stacking operation is the easiest way to construct new topological phases from old ones.

The most simple case is when there is no symmetry, and we allow any local interactions between layers. We denote such stacking operation by $\boxtimes$ (indeed it is related to the Deligne tensor product and we do not need to distinguish the notations). Obviously, it is commutative and associative, $\mathcal{C} \boxtimes \mathcal{D} = \mathcal{D} \boxtimes \mathcal{C}, (\mathcal{C}_1 \boxtimes \mathcal{C}_2) \boxtimes \mathcal{C}_3 = \mathcal{C}_1 \boxtimes (\mathcal{C}_2 \boxtimes \mathcal{C}_3)$. The trivial phase $\mathbf{1}$ (tensor product states) is the identity, $\mathcal{C} \boxtimes \mathbf{1} = \mathbf{1} \boxtimes \mathcal{C}$. Therefore, topological phases form a commutative monoid (a "semi-group" that requires the existence of identity but not inverse) under stacking.

When stacking two systems $\mathcal{C}, \mathcal{D}$ with the symmetries G_1, G_2, there is a choice for the new symmetry of the two-layer system, that puts restrictions on what *symmetric* interactions between layers can be added. One natural choice is $G_1 \times G_2$, denoted by $\mathcal{C} \boxtimes \mathcal{D}$, that is to preserve the symmetry of each layer respectively.

When $G_1 = G_2 = G$, another natural choice for the new symmetry is G, denoted by $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ (recall that $\mathcal{E}$ is the representation category of G) where G is viewed as a subgroup of $G \times G$ via the embedding $g \mapsto (g, g)$. In other words, for the stacking $\boxtimes_{\mathcal{E}}$ we allow the inter-layer interactions that preserve only the subgroup G. $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D}$ share the same symmetry G. Therefore, topological phases with symmetry G again form a commutative monoid under the stacking $\boxtimes_{\mathcal{E}}$ which preserves the symmetry.

A topological phase $\mathcal{C}$ with symmetry $\mathcal{E}$ is called invertible if there exists another phase $\mathcal{D}$ such that $\mathcal{C} \boxtimes_{\mathcal{E}} \mathcal{D} = \mathbf{1}$. In this case $\mathcal{C}$ and $\mathcal{D}$ are time-reversal conjugates. All invertible topological phases with symmetry $\mathcal{E}$ form an Abelian group $\mathbf{Inv}_{\mathcal{E}}$ under stacking. The chiral central charges of the edge states add up under stacking, so taking the central charge is a group homomorphism from invertible phases $\mathbf{Inv}_{\mathcal{E}}$ to $\mathbb{Q}$. Its image is $c_{\mathcal{E}}^{\min}\mathbb{Z}$, where $c_{\mathcal{E}}^{\min}$ is the smallest positive central charge. From this point of view, the non-chiral invertible phases (the kernel of the above group homomorphism) are the symmetry protected topological (SPT) phases:

$$0 \to \mathrm{SPT}_{\mathcal{E}} \to \mathbf{Inv}_{\mathcal{E}} \to c_{\mathcal{E}}^{\min}\mathbb{Z} \to 0,$$

Since $H^2(\mathbb{Z}, M) = 0$ for any abelian group M, the above must be a trivial extension, namely

$$\text{Invertible topological phases with symmetry} \cong \mathrm{SPT}_{\mathcal{E}} \times c_{\mathcal{E}}^{\min}\mathbb{Z},$$

For boson systems, $c_{\mathrm{Rep}(G)}^{\min} = 8$ corresponds to the E_8 state. For fermion systems with symmetry $G^f = G_b \times \mathbb{Z}_2^f$, $c_{\mathrm{sRep}(G_b \times \mathbb{Z}_2^f)}^{\min} = 1/2$ corresponds to the $p + \mathrm{i}p$ superconducting state. For other fermionic symmetries it will be clear how to determine $c_{\mathcal{E}}^{\min}$ at the end of this section.

Invertible phases do not support any non-trivial quasiparticle statistics. For the non-invertible topological phases, we have to seriously study their quasiparticle excitations, characterized by $\mathcal{E} \hookrightarrow \mathcal{C} \hookrightarrow \mathcal{M}$.

Again, consider the stacking operation corresponding to the no-symmetry case firstly. It is given by the Deligne tensor product $\boxtimes$, which defines a monoidal structure on the 2-category of unitary braided fusion categories (more generally, of Abelian categories). For two UMTCs $\mathcal{C}, \mathcal{D}$, $\mathcal{C} \boxtimes \mathcal{D}$ is still a UMTC. (By construction, $\mathrm{Hom}_{\mathcal{C}\boxtimes\mathcal{D}}(A \boxtimes B, X \boxtimes Y) = \mathrm{Hom}_{\mathcal{C}}(A, X) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(B, Y)$. All the structures follows component-wise.) There is a parallel story for $\mathrm{UMTC}_{/\mathcal{E}}$, a monoidal structure $\boxtimes_{\mathcal{E}}$ such that the "stacking" of two $\mathrm{UMTC}_{/\mathcal{E}}$s is still a $\mathrm{UMTC}_{/\mathcal{E}}$. We introduce this construction and generalize it to modular extensions. Such stacking operation is for $\mathrm{UMTC}_{/\mathcal{E}}$ together with their modular extensions, thus physically the stacking operations for topological phases with symmetry $\mathcal{E}$.

The basic idea is to first construct $\mathcal{C} \boxtimes \mathcal{D}$ which has symmetry $\mathcal{E} \boxtimes \mathcal{E}$, and then break the symmetry down to $\mathcal{E}$. To do this, we construct a canonical condensable algebra $L_\mathcal{C}$ in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ for any UBFC $\mathcal{C}$. In particular, $L_\mathcal{E}$ is the algebra that corresponds to the symmetry breaking $\mathcal{E} \boxtimes \mathcal{E} \to \mathcal{E}$.

Let $\mathcal{C}$ be a braided fusion category and $\mathcal{A}$ a fusion category, a tensor functor $F : \mathcal{C} \to \mathcal{A}$ is called a central functor if it factorizes through $Z(\mathcal{A})$, i.e., there exists a braided tensor functor $F' : \mathcal{C} \to Z(\mathcal{A})$ such that $F = \mathrm{for}_\mathcal{A} F'$, where $\mathrm{for}_\mathcal{A}$ is the forgetful tensor functor $\mathrm{for}_\mathcal{A} : Z(\mathcal{A}) \to \mathcal{A}$, $(X, b_{X,-}) \mapsto X$ that forgets the half-braidings.

Lemma 11.1 (DMNO [4])**.** Let $F : \mathcal{C} \to \mathcal{A}$ be a central functor, and $R : \mathcal{A} \to \mathcal{C}$ the right adjoint functor of F. Then the object $A = R(\mathbf{1}) \in \mathcal{C}$ has a canonical structure of condensable algebra. $\mathcal{C}_A$ is monoidally equivalent to the image of F, i.e. the smallest fusion subcategory of $\mathcal{A}$ containing $F(\mathcal{C})$.

If $\mathcal{C}$ is a unitary braided fusion category, it is naturally embedded into $Z(\mathcal{C})$, by taking $X \mapsto (X, b_{X,-} = c_{X,-})$. So is $\overline{\mathcal{C}}$. Therefore, we have a braided tensor functor $\mathcal{C} \boxtimes \overline{\mathcal{C}} \to Z(\mathcal{C})$. Compose it with the forgetful functor $\mathrm{for}_\mathcal{C} : Z(\mathcal{C}) \to \mathcal{C}$ we get a central functor

$$\begin{aligned} \otimes : \mathcal{C} \boxtimes \overline{\mathcal{C}} &\to \mathcal{C} \\ X \boxtimes Y &\mapsto X \otimes Y. \end{aligned}$$

Let R be its right adjoint functor, we obtain a condensable algebra $L_\mathcal{C} := R(\mathbf{1}) \cong \oplus_i (i \boxtimes i^*) \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ and $\mathcal{C} = (\mathcal{C} \boxtimes \overline{\mathcal{C}})_{L_\mathcal{C}}$, $\dim(L_\mathcal{C}) = \dim(\mathcal{C})$. In particular, for a symmetric category $\mathcal{E}$, $L_\mathcal{E}$ is a condensable algebra in $\mathcal{E} \boxtimes \mathcal{E}$, and $\mathcal{E} = (\mathcal{E} \boxtimes \mathcal{E})_{L_\mathcal{E}} = (\mathcal{E} \boxtimes \mathcal{E})^0_{L_\mathcal{E}}$ for $\mathcal{E}$ is symmetric, all $L_\mathcal{E}$-modules are local.

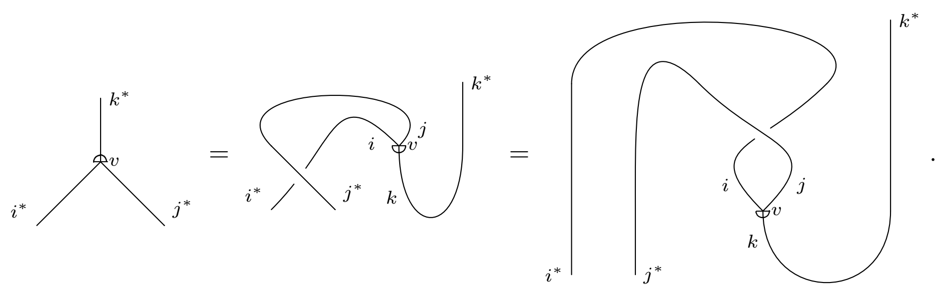
Exercise 11.1. The structure coefficients of $L_\mathcal{C}$ is

$$M^{k \boxtimes k^*, u \boxtimes v}_{i \boxtimes i^*, j \boxtimes j^*} = \delta_{uv} \sqrt{\frac{1}{\dim \mathcal{C}} \frac{d_i d_j}{d_k}}, \tag{185}$$

provided that we choose the basis vertex $p^{k,u}_{ij} \otimes_\mathbb{C} p^{k^*,v}_{i^* j^*}$, where

$$p^{k^*,v}_{i^* j^*} = \left(\left(p^{k,v}_{ij} \right)^\dagger \right)^* \overline{c}_{i^*, j^*} = \left(\overline{c}_{i,j} \left(p^{k,v}_{ij} \right)^\dagger \right)^*, \tag{186}$$

or graphically



(187)

Here $p_{ij}^{k,u}$ is orthonormal, $p_{ij}^{k,u}\left(p_{ij}^{k,v}\right)^\dagger = \delta_{uv}\,\mathrm{id}_k$, and $\bar{c}_{i,j}$ is the braiding in $\overline{\mathcal{C}}$, $\bar{c}_{i,j} = c_{j,i}^{-1} = c_{j,i}^\dagger$. Try to verify the conditions (173)(176)(177) and convince yourself that $L_\mathcal{C}$ is indeed a condensable algebra.

Now, we are ready to define the stacking operation for $\mathrm{UMTC}_{/\mathcal{E}}$'s as well as their modular extensions.

Definition 11.1. Let $\mathcal{C},\mathcal{D}$ be $\mathrm{UMTC}_{/\mathcal{E}}$'s, and $\mathcal{M}_\mathcal{C},\mathcal{M}_\mathcal{D}$ their modular extensions. The stacking is defined by:

$$\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D} := (\mathcal{C}\boxtimes\mathcal{D})^0_{L_\mathcal{E}},\quad \mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D} := (\mathcal{M}_\mathcal{C}\boxtimes\mathcal{M}_\mathcal{D})^0_{L_\mathcal{E}} \tag{188}$$

Theorem 11.2. $\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D}$ is a $\mathrm{UMTC}_{/\mathcal{E}}$, and $\mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D}$ is a modular extension of $\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D}$.

Proof. The embeddings $\mathcal{E} = (\mathcal{E}\boxtimes\mathcal{E})^0_{L_\mathcal{E}} \hookrightarrow (\mathcal{C}\boxtimes\mathcal{D})^0_{L_\mathcal{E}} = \mathcal{C}\boxtimes_\mathcal{E}\mathcal{D} \hookrightarrow \mathcal{E}'|_{\mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D}} \hookrightarrow (\mathcal{M}_\mathcal{C}\boxtimes\mathcal{M}_\mathcal{D})^0_{L_\mathcal{E}} = \mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D}$ are obvious. So $\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D}$ is a UBFC over $\mathcal{E}$. Also

$$\dim(\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D}) = \frac{\dim(\mathcal{C}\boxtimes\mathcal{D})}{\dim(L_\mathcal{E})} = \frac{\dim(\mathcal{C})\dim(\mathcal{D})}{\dim(\mathcal{E})}, \tag{189}$$

and $\mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D}$ is a UMTC,

$$\dim(\mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D}) = \frac{\dim(\mathcal{M}_\mathcal{C}\boxtimes\mathcal{M}_\mathcal{D})}{\dim(L_\mathcal{E})^2} = \dim(\mathcal{C})\dim(\mathcal{D}). \tag{190}$$

Therefore, $\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D}$ and $\mathcal{E}'|_{\mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D}}$ have the same total quantum dimension, thus by Lemma 9.1 we know that they are the same. By Remark 10, $\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D}$ is a $\mathrm{UMTC}_{/\mathcal{E}}$, and $\mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{D}$ is a modular extension of $\mathcal{C}\boxtimes_\mathcal{E}\mathcal{D}$. □

Note that $\mathcal{C}\boxtimes_\mathcal{E}\mathcal{E} = \mathcal{C}$. Therefore, for any modular extension $\mathcal{M}_\mathcal{E}$ of $\mathcal{E}$, $\mathcal{M}_\mathcal{C}\boxtimes_\mathcal{E}\mathcal{M}_\mathcal{E}$ is still a modular extension of $\mathcal{C}$. Physically this means that stacking with an invertible phase will not change the bulk excitations. In the following we want to show the inverse, that one can extract the "difference", a modular extension of $\mathcal{E}$, or an invertible phase, between two modular extensions of $\mathcal{C}$.

Lemma 11.3. We have $(\mathcal{C}\boxtimes\overline{\mathcal{C}})^0_{L_\mathcal{C}} = \mathcal{C}'|_\mathcal{C}$.

Proof. $(\mathcal{C}\boxtimes\overline{\mathcal{C}})_{L_\mathcal{C}}$ is equivalent to $\mathcal{C}$ (as a fusion category). Moreover, for $X\in\mathcal{C}$ the equivalence gives the free module $(X\boxtimes\mathbf{1})\otimes L_\mathcal{C} \cong (\mathbf{1}\boxtimes X)\otimes L_\mathcal{C}$. $(X\boxtimes\mathbf{1})\otimes L_\mathcal{C}$ is a local $L_\mathcal{C}$ module if and only if $X\boxtimes\mathbf{1}$ centralize $L_\mathcal{C}$. This is the same as $X\in\mathcal{C}'|_\mathcal{C}$. Therefore, we have $(\mathcal{C}\boxtimes\overline{\mathcal{C}})^0_{L_\mathcal{C}} = \mathcal{C}'|_\mathcal{C}$. □

Lemma 11.4 (FFRS [10])**.** For a non-commutative algebra A, we have the left center A_l of A, with algebra embedding $e_l : A_l \to A$, which is the maximal subalgebra such that $m(\mathrm{id}_A\otimes e_l)c_{A_l,A} = m(e_l\otimes\mathrm{id}_A)$. Similarly the right center A_r with $e_r : A_r \to A$, is the maximal subalgebra such that $m(e_r\otimes\mathrm{id}_A)c_{A,A_r} = m(\mathrm{id}_A\otimes e_r)$. A_l and A_r are commutative subalgebras, thus condensable. There is a canonical equivalence between the categories of local modules over the left and right centers, $\mathcal{C}^0_{A_l} = \mathcal{C}^0_{A_r}$.

Theorem 11.5. let $\mathcal{M}$ and $\mathcal{M}'$ be two modular extensions of the UMTC$_{/\mathcal{E}}$ $\mathcal{C}$. There exists a unique $\mathcal{K} \in \mathcal{M}_{ext}(\mathcal{E})$ such that $\mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M} = \mathcal{M}'$. Such $\mathcal{K}$ is given by

$$\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{C}}}. \tag{191}$$

Proof. $\mathcal{K}$ is a modular extension of $\mathcal{E}$. This follows Lemma 11.3, that $\mathcal{E} = \mathcal{C}'|_{\mathcal{C}} = (\mathcal{C} \boxtimes \overline{\mathcal{C}})^0_{L_{\mathcal{C}}}$ is a full subcategory of $\mathcal{K}$. $\mathcal{K}$ is a UMTC by construction, and $\dim(\mathcal{K}) = \frac{\dim(\mathcal{M})\dim(\mathcal{M}')}{\dim(L_{\mathcal{C}})^2} = \dim(\mathcal{E})^2$.

To show that $\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})_{L_{\mathcal{C}}}$ satisfies $\mathcal{M}' = \mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M}$, note that $\mathcal{M}' = \mathcal{M}' \boxtimes \mathbf{Hilb} = \mathcal{M}' \boxtimes (\overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{L_{\overline{\mathcal{M}}}}$. It suffices that

$$\begin{aligned}(\mathcal{M}' \boxtimes \overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{\mathbf{1} \boxtimes L_{\overline{\mathcal{M}}}} &= [(\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{C}}} \boxtimes \mathcal{M}]^0_{L_{\mathcal{E}}} \\ &= (\mathcal{M}' \boxtimes \overline{\mathcal{M}} \boxtimes \mathcal{M})^0_{(L_{\mathcal{C}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{E}})}.\end{aligned} \tag{192}$$

While $\mathbf{1} \boxtimes L_{\overline{\mathcal{M}}}$ and $(L_{\mathcal{C}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{E}})$ turns out to be left and right centers of the algebra $(L_{\mathcal{C}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\overline{\mathcal{M}}})$.

If $\mathcal{M}' = \mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M} = (\mathcal{K} \boxtimes \mathcal{M})^0_{L_{\mathcal{E}}}$, then

$$\begin{aligned}\mathcal{K} = (\mathcal{K} \boxtimes \mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{\mathbf{1} \boxtimes L_{\mathcal{M}}} &= (\mathcal{K} \boxtimes \mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{(L_{\mathcal{E}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{C}})} \\ &= [(\mathcal{K} \boxtimes_{\mathcal{E}} \mathcal{M}) \boxtimes \overline{\mathcal{M}}]^0_{L_{\mathcal{C}}} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{C}}}.\end{aligned} \tag{193}$$

It is similar here that $\mathbf{1} \boxtimes L_{\mathcal{M}}$ and $(L_{\mathcal{E}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{C}})$ are the left and right centers of the algebra $(L_{\mathcal{E}} \boxtimes \mathbf{1}) \otimes (\mathbf{1} \boxtimes L_{\mathcal{M}})$. This proves the uniqueness of $\mathcal{K}$.

The above established the equivalences between UMTCs. To further show that they are equivalences between modular extensions, one need to check the embeddings of $\mathcal{E}, \mathcal{C}$. Here the only non-trivial braided tensor equivalences are those between the categories of local modules over left and right centers. By the detailed construction given in Ref. [10], one can check that they indeed preserve the embeddings of $\mathcal{E}, \mathcal{C}$. □

Let us list several consequences of Theorem 11.5.

Corollary 11.6. $\mathcal{M}_{ext}(\mathcal{E})$ forms a finite Abelian group. The identity is $Z(\mathcal{E})$ and the inverse of $\mathcal{M}$ is $\overline{\mathcal{M}}$.

Proof. It is easy to verify that the stacking $\boxtimes_{\mathcal{E}}$ for modular extensions is associative and commutative. To show that they form a group we only need to find out the identity and inverse. In this case $\mathcal{K} = (\mathcal{M}' \boxtimes \overline{\mathcal{M}})^0_{L_{\mathcal{E}}} = \mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}}$, Theorem 11.5 becomes $\mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M} = \mathcal{M}'$, for any modular extensions $\mathcal{M}, \mathcal{M}'$ of $\mathcal{E}$. Thus, $\overline{\mathcal{M}'} \boxtimes_{\mathcal{E}} \mathcal{M}' = \overline{\mathcal{M}'} \boxtimes_{\mathcal{E}} \mathcal{M}' \boxtimes_{\mathcal{E}} \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M} = \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M}$, i.e. $\mathcal{Z}_{\mathcal{E}} := \overline{\mathcal{M}} \boxtimes_{\mathcal{E}} \mathcal{M}$ is the same category for any extension $\mathcal{M}$, which is exactly the identity element. It is then obvious that the inverse of $\mathcal{M}$ is $\overline{\mathcal{M}}$. The finiteness follows from Ref. [1].

In fact, the identity $\mathcal{Z}_{\mathcal{E}}$ should be $Z(\mathcal{E})$, the Drinfeld center of $\mathcal{E}$. (This is Theorem 11.8. The embedding $\mathcal{E} \hookrightarrow Z(\mathcal{E})$ is given by the lift of the identity functor on $\mathcal{E}$, i.e., $\mathcal{E} \hookrightarrow Z(\mathcal{E}) \to \mathcal{E}$ equals $\mathrm{id}_{\mathcal{E}}$.) □

Example 11.1. $\mathcal{M}_{ext}(\text{sRep}(\mathbb{Z}_2^f)) \cong \mathbb{Z}_{16}$, with central charge $c = n/2 \mod 8, n = 0, 1, 2, \ldots, 15$. This is the 16-fold way [11].

Example 11.2 (LKW [14])**.** $\mathcal{M}_{ext}(\text{Rep}(G)) \cong H^3(G, U(1))$, all with central charge $c = 0 \mod 8$. This agrees with the classification of bosonic SPT phases [3].

Corollary 11.7. For a $\text{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, $\mathcal{M}_{ext}(\mathcal{C})$, if exists, forms a $\mathcal{M}_{ext}(\mathcal{E})$-torsor. The action of $\mathcal{M}_{ext}(\mathcal{E})$ on $\mathcal{M}_{ext}(\mathcal{C})$ is given by the stacking $\boxtimes_{\mathcal{E}}$.

Below is a standalone theorem that fixes the unit element in the Abelian group of modular extensions.

Theorem 11.8. Let $\mathcal{M}$ be a modular extension of a $\text{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$:

$$(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}} = Z(\mathcal{E}). \tag{194}$$

In particular, this means that $\mathcal{Z}_\mathcal{E} = Z(\mathcal{E})$.

Proof. There is a Lagrangian algebra $L_\mathcal{M}$ in $\mathcal{M} \boxtimes \overline{\mathcal{M}}$, such that the category of $L_\mathcal{M}$-modules in $\mathcal{M} \boxtimes \overline{\mathcal{M}}$ is $(\mathcal{M} \boxtimes \overline{\mathcal{M}})_{L_\mathcal{M}} = \mathcal{M}$, via the functor $(i \boxtimes \mathbf{1}) \otimes L_\mathcal{M} \mapsto i$. $L_\mathcal{M}$ is a condensable algebra over $L_\mathcal{C}$, and also a condensable algebra in $(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}$. We would like to show that $[(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}]_{L_\mathcal{M}} = \mathcal{E}$. To see this, note that $\mathcal{E} \hookrightarrow (\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}$, the image of $\mathcal{E}$ identifies with the free $L_\mathcal{C}$-modules $(i \boxtimes \mathbf{1}) \otimes L_\mathcal{C}, i \in \mathcal{E}$. Further check the free $L_\mathcal{M}$-modules in $(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}$ generated by these objects, and we find that $[(i \boxtimes \mathbf{1}) \otimes L_\mathcal{C}] \otimes_{L_\mathcal{C}} L_\mathcal{M} \cong (i \boxtimes \mathbf{1}) \otimes L_\mathcal{M} \mapsto i$. This means that $\mathcal{E} \subset [(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}]_{L_\mathcal{M}}$. Since they have the same total quantum dimension, we must have $[(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}]_{L_\mathcal{M}} = \mathcal{E}$. Since $L_\mathcal{M}$ is Lagrangian in $(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}$, $(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}} = Z([(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}]_{L_\mathcal{M}}) = Z(\mathcal{E})$. Moreover, $- \otimes_{L_\mathcal{C}} L_\mathcal{M} : (\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}} \to [(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}]_{L_\mathcal{M}}$ coincides with the forgetful functor $Z(\mathcal{E}) \to \mathcal{E}$. Thus the embedding $\mathcal{E} \hookrightarrow (\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}}$ composed with the forgetful functor $Z(\mathcal{E}) \to \mathcal{E}$ gives the identity functor on $\mathcal{E}$. □

Note that in the proof we actually did not use the fact that $\mathcal{E}$ is a symmetric category, indeed

Corollary 11.9. Let $\mathcal{M}$ a modular tensor category and $\mathcal{C}$ a fusion subcategory of $\mathcal{M}$:

$$(\mathcal{M} \boxtimes \overline{\mathcal{M}})^0_{L_\mathcal{C}} = Z(\mathcal{C}'|_\mathcal{M}). \tag{195}$$

We conclude the main results. Topological phase with symmetry $\mathcal{E}$ are classified by the triple $(\mathcal{C}, \mathcal{M}, c)$. We mathematically constructed the stacking operation between them,

$$(\mathcal{C}_1, \mathcal{M}_1, c_1) \boxtimes_\mathcal{E} (\mathcal{C}_2, \mathcal{M}_2, c_2) = (\mathcal{C}_1 \boxtimes_\mathcal{E} \mathcal{C}_2, \mathcal{M}_1 \boxtimes_\mathcal{E} \mathcal{M}_2, c_1 + c_2). \tag{196}$$

In particular, the trivial phase with symmetry $\mathcal{E}$ is given by $(\mathcal{E}, Z(\mathcal{E}), c = 0)$, and invertible topological phases with symmetry $\mathcal{E}$ are described by $(\mathcal{E}, \mathcal{M}, c)$, where $\mathcal{M}$ is a modular extension of $\mathcal{E}$, $\mathcal{M} \in \mathcal{M}_{ext}(\mathcal{E})$. They indeed form a Abelian group under the stacking operation defined above. For boson systems, $\mathcal{E} = \text{Rep}(G)$, $\mathcal{M}_{ext}(\text{Rep}(G)) \cong H^3(G, U(1))$, and they all have central charge $c = 0 \mod 8$. The group structure $H^3(G, U(1)) \times 8\mathbb{Z}$ is recovered. For fermion systems, we expect that $\mathcal{M}_{ext}(\text{sRep}(G^f))$ gives a full classification of invertible phases. We can obtain both the fermionic SPT, namely the $c = 0$ part in $\mathcal{M}_{ext}(\text{sRep}(G^f))$, and the smallest positive central charge $c^{\min}_{\text{sRep}(G^f)}$ of the chiral invertible phases. Thus, invertible topological phases with symmetry $\mathcal{E}$ are classified by

$$\text{SPT}_{\mathcal{E}} \times c^{\min}_{\mathcal{E}}\mathbb{Z}, \quad (\text{SPT}_{\mathcal{E}} \times c^{\min}_{\mathcal{E}}\mathbb{Z})/8\mathbb{Z} \cong \mathcal{M}_{ext}(\mathcal{E}). \tag{197}$$

Also if we stack an invertible phase $(\mathcal{E}, \mathcal{M}_{\mathcal{E}}, c_1)$ onto $(\mathcal{C}, \mathcal{M}, c_2)$, it only changes the modular extension part,

$$(\mathcal{E}, \mathcal{M}_{\mathcal{E}}, c_1) \boxtimes_{\mathcal{E}} (\mathcal{C}, \mathcal{M}, c_2) = (\mathcal{C}, \mathcal{M}_{\mathcal{E}} \boxtimes_{\mathcal{E}} \mathcal{M}, c_1 + c_2). \tag{198}$$

By stacking all invertible phases (all modular extensions of $\mathcal{E}$), all modular extensions of $\mathcal{C}$ can be generated. Moreover, the "difference" between two modular extensions is a unique invertible phase (unique modular extension of $\mathcal{E}$). In short, the modular extensions of $\text{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$ form a torsor over the Abelian group $\mathcal{M}_{ext}(\mathcal{E})$.

Therefore, a $\text{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$, if its modular extension exists, already fixed the topological phase up to invertible ones. Appending the modular extension to the label further fixes the invertible ones up to E_8 states[1], and appending the central charge c totally fixes the topological phase. On the other hand, if a $\text{UMTC}_{/\mathcal{E}}$ $\mathcal{C}$ has no modular extension, namely the symmetry can not be gauged, it is anomalous and can only be realized on the boundary of (3+1)D SPT phases [2].

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